Chapter 2

Numerical Method

2.1 Introduction

In the present chapter the code AERO, used in the present study, is described. The code permits to solve the Euler equations, the Navier Stokes equations for laminar flows and to use different turbulence models for RANS, LES and hybrid RANS/LES approaches. The unknown quantities are the density, the components of the momentum and the total energy per unit volume. AERO employs a mixed finite-volume/finite-element formulation for the spatial discretization of the equations. Finite-volumes are used for the convective fluxes and finite-elements (P1) for the diffusive ones. The resulting scheme is second order accurate in space. The equations can be advanced in time with explicit low-storage Runge-Kutta schemes. Also implicit time advancing is possible, based on a linearised method that is second order accurate in time.
2.2 Set of equations

In the AERO code the Navier Stokes equations are numerically normalised with the following reference quantities:

- $L_{ref} \implies$ characteristic length of the flow
- $U_{ref} \implies$ velocity of the free-stream flow
- $\rho_{ref} \implies$ density of the free-stream flow
- $\mu_{ref} \implies$ molecular viscosity of the free-stream flow

The flow variables can be normalised with the reference quantities as follows:

$$
\rho^* = \frac{\rho}{\rho_{ref}} \quad u_j^* = \frac{u_j}{U_{ref}} \quad p^* = \frac{p}{p_{ref}}
$$

$$
E^* = \frac{E}{\rho_{ref} U_{ref}^2} \quad \mu^* = \frac{\mu}{\mu_{ref}} \quad t^* = t \frac{L_{ref}}{U_{ref}}. \quad (2.1)
$$

The non-dimensional form of the Navier Stokes equations can be obtained substituting the reference quantities Eq. (2.1) in the set of equations described in Eq. (1.1). As an example the non-dimensional equations for the laminar case are reported in the following:

$$
\begin{align*}
\frac{\partial \rho^*}{\partial t^*} + \frac{\partial (\rho^* u_j^*)}{\partial x_j^*} &= 0 \\
\frac{\partial (\rho^* u_j^*)}{\partial t^*} + \frac{\partial (\rho^* u_i^* u_j^*)}{\partial x_i^*} &= -\frac{\partial p^*}{\partial x_i^*} + \frac{1}{Re} \frac{\partial \sigma_{ij}^*}{\partial x_j^*} \\
\frac{\partial (\rho^* E^*)}{\partial t^*} + \frac{\partial (\rho^* E^* u_j^*)}{\partial x_j^*} &= -\frac{\partial (p^* u_j^*)}{\partial x_j^*} + \frac{1}{Re} \frac{\partial (u_j^* \sigma_{ij}^*)}{\partial x_i^*} - \frac{\gamma}{Re Pr} \frac{\partial}{\partial x_j^*} \left[ \mu^* \left( E^* - \frac{1}{2} u_j^* u_j^* \right) \right]
\end{align*}
\quad (2.2)
$$
where the Reynolds number, \( Re = \frac{U_{ref} L_{ref}}{\nu} \), is based on the references quantities, \( U_{ref} \) and \( L_{ref} \), the Prandtl number, \( Pr \), can be assumed constant for a gas and equal to:

\[
Pr = \frac{C_p \mu}{k}
\]

and \( \gamma = C_p/C_v \) is the ratio between the specific heats at constant pressure and volume. Also the constitutive equations for the viscous stresses and the state equations may be written in non-dimensional form as follows:

\[
\sigma_{ij}^* = -\frac{2}{3} \mu^* \left( \frac{\partial u_i^*}{\partial x_k} \delta_{ij} \right) + \mu^* \left( \frac{\partial u_i^*}{\partial x_j} + \frac{\partial u_j^*}{\partial x_i} \right)
\]

\[
p^* = (\gamma - 1) \rho^* \left( E^* - \frac{1}{2} u_j^* u_j^* \right) .
\]  \hspace{1cm} (2.3)

In order to rewrite the governing equations in a compact form more suitable for the discrete formulation, the following unknown variables are grouped together in the \( \mathbf{W} \) vector:

\[
\mathbf{W} = (\rho, \rho u, \rho v, \rho w, \rho E)^T.
\]

If two other vectors, \( \mathbf{F} \) and \( \mathbf{V} \) are defined as function of \( \mathbf{W} \), as follows:

\[
\mathbf{F} = \begin{pmatrix}
\rho u & \rho v & \rho w \\
\rho u^2 + p & \rho u v & \rho u w \\
\rho u w & \rho v^2 + p & \rho v w \\
\rho u w & \rho v w & \rho w^2 + p
\end{pmatrix}
\]

and

\[
\mathbf{V} = \begin{pmatrix}
0 & 0 & 0 \\
\sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\
\sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\
\sigma_{xz} & \sigma_{yz} & \sigma_{zz}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\rho \sigma_{xx} + v \sigma_{xy} + w \sigma_{xz} - q_x \\
\rho \sigma_{xy} + \rho \sigma_{yy} + w \sigma_{yz} - q_y \\
\rho \sigma_{xz} + v \sigma_{yz} + w \sigma_{zz} - q_z
\end{pmatrix}
\]
\[
\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_j} F_j(W) - \frac{1}{Re} \frac{\partial}{\partial x_j} V_j(W, \nabla W) = 0. \tag{2.4}
\]

It is important to stress that the vectors \(F\) and \(V\) are respectively the convective fluxes and the diffusive fluxes.

### 2.3 Space discretization

Spatial discretization is based on a mixed finite-volume/finite-element formulation. A finite volume upwind formulation is used for the treatment of the convective fluxes while a classical Galerkin finite-element centred approximation is employed for the diffusive terms.

The computational domain \(\Omega\) is approximated by a polygonal domain \(\Omega_h\). This polygonal domain is then divided in \(N_t\) tetrahedral elements \(T_i\) by a standard finite-element triangulation process:

\[
\Omega_h = \bigcup_{i=1}^{N_t} T_i. \tag{2.5}
\]

The set of elements \(T_i\) forms the grid used in the finite-element formulation. The dual finite-volume grid can be built starting from the triangulation following two ways: the Barth cell construction (BC) or the medians method. The first one is useful for significantly stretched grids. This type of cells can be obtained in 3D as follows: to build the cell centred at node \(i\), let us consider all the neighbouring nodes of \(i\) \((j)\). For each element containing the nodes \(i\) and \(j\), the cell surface is given by the triangles connecting the middle of the edge joining these two vertexes, the surface centre of the faces of the element having this edge in common, and the volume centre of the element. The surface centre of a given face is the centre of its circumscribed circle, if the face comprises only acute angles, otherwise it is the middle of its longest edge, and the volume centre of an element is the centre of its circumscribed sphere if the former is located inside the element, otherwise, it is the surface centre (among those of the four tetrahedron faces), which is closest to the
centre of the circumscribed sphere. Although the BC can be built starting from a generic tetrahedrization, it is interesting to consider the case of a Cartesian mesh, thus, made of rectangle parallelepipeds (thereafter called bricks), which are cut in a particular way in tetrahedrons, following [9]. This division splits each brick in six identical tetrahedra, each being the mirror image of its neighbours (called also English flag division see Fig. 2.1(a)). Starting from such a tetrahedization, the BC cells are bricks, centred around the vertices’s of the mesh, as can be seen in Fig. 2.1b, in which the trace of the division of an element into BC is shown.

![Figure 2.1. New finite-volume cells in 3D: (a) division in tetrahedrons, (b) trace of BC on a tetrahedron resulting from the previous division.](image)

In the medians method a finite-volume cell is constructed around each node $a_i$ of the triangulation, dividing in 4 sub-tetrahedra every tetrahedron having $a_i$ as a vertex by means of the median planes. $C_i$ is the union of the resulting sub-tetrahedra having $a_i$ as a vertex and they have the following property:

$$\Omega_h = \bigcup_{i=1}^{N_c} C_i.$$  

(2.6)

where $N_c$ is the number of cells, which is equal to the number of the nodes of the triangulation.

**Convective fluxes**

Indicating the basis functions for the finite-volume formulation as follows:
\[ \psi^{(i)}(P) = \begin{cases} 1 & \text{if } P \in C_i \\ 0 & \text{otherwise} \end{cases} \]

the Galerkin formulation for the convective fluxes is obtained by multiplying the convective terms of (2.4) by the basis function \( \psi^{(i)} \), integrating on the domain \( \Omega_h \) and using the divergence theorem. In this way the results are:

\[
\int \int_{\Omega_h} \left( \frac{\partial F_j}{\partial x_j} \right) \psi^{(i)} \, dx \, dy = \int \int_{C_i} \frac{\partial F_j}{\partial x_j} \, d\Omega = \int_{\partial C_i} F_j n_j \, d\sigma
\]

where \( d\Omega, \, d\sigma \) and \( n_j \) are the elementary measure of the cell, of its boundary and the \( j \)th component of the normal external to the cell \( C_i \) respectively.

The total contribution to the convective fluxes is:

\[
\sum_j \int_{\partial C_{ij}} \mathcal{F}(W, \vec{n}) \, d\sigma
\]

where \( j \) are all the neighbouring nodes of \( i \), \( \mathcal{F}(W, \vec{n}) = F_j(W)n_j \), \( \partial C_{ij} \) is the boundary between cells \( C_i \) and \( C_j \), and \( \vec{n} \) is the outer normal to the cell \( C_i \).

The basic component for the approximation of the convective fluxes is the Roe scheme, Ref. [10]:

\[
\int_{\partial C_{ij}} \mathcal{F}(W, \vec{n}) \, d\sigma \simeq \Phi^R(W_i, W_j, \vec{v}_{ij})
\]

where

\[
\vec{v}_{ij} = \int_{\partial C_{ij}} \vec{n} \, d\sigma
\]

and \( W_k \) is the solution vector at the \( k \)-th node of the discretization.

The numerical fluxes, \( \Phi^R \), are evaluated as follows:

\[
\Phi^R(W_i, W_j, \vec{v}_{ij}) = \underbrace{\mathcal{F}(W_i, \vec{v}_{ij}) + \mathcal{F}(W_j, \vec{v}_{ij})}_{\text{centered}} - \underbrace{\gamma_s d^R(W_i, W_j, \vec{v}_{ij})}_{\text{upwinding}}
\]
where $\gamma_s \in [0,1]$ is a parameter which directly controls the upwinding of the scheme and

$$d^R(W_i, W_j, \bar{\nu}_{ij}) = \left| \mathcal{R}(W_i, W_j, \bar{\nu}_{ij}) \right| \frac{W_j - W_i}{2}. \quad (2.7)$$

$\mathcal{R}$ is the Roe matrix and is defined as:

$$\mathcal{R}(W_i, W_j, \bar{\nu}_{ij}) = \frac{\partial \mathcal{F}}{\partial \bar{W}}(\bar{W}, \bar{\nu}_{ij}) \quad (2.8)$$

where $\bar{W}$ is the Roe average between $W_i$ and $W_j$.

The classical Roe scheme is obtained as a particular case by imposing $\gamma_s = 1$. The accuracy of this scheme is only 1st order. In order to increase the order of accuracy of the scheme the MUSCL (Monotone Upwind Schemes for Conservation Laws) reconstruction method, introduced by Van Leer, Ref. [11], is employed. This method expresses the Roe flux as a function of the extrapolated values of $\bar{W}$ at the interface between the two cells $C_i$ and $C_j$, $W_{ij}$ and $W_{ji}$:

$$\int_{\partial C_{ij}} \mathcal{F}(W, \bar{\nu}) \, d\sigma \simeq \Phi^R(W_{ij}, W_{ji}, \bar{\nu}_{ij})$$

where $W_{ij}$ and $W_{ji}$ are defined as follows:

$$W_{ij} = W_i + \frac{1}{2} (\bar{\nabla}W)_{ij} \cdot \bar{\nu}_{ij}, \quad (2.9)$$

$$W_{ji} = W_j + \frac{1}{2} (\bar{\nabla}W)_{ji} \cdot \bar{\nu}_{ij}. \quad (2.10)$$

To estimate the gradients $(\bar{\nabla}W)_{ij} \cdot \bar{\nu}_{ij}$ and $(\bar{\nabla}W)_{ji} \cdot \bar{\nu}_{ij}$ the V6 scheme is used, Ref. [12]:
\begin{equation}
(\vec{\nabla}W)_{ij} \cdot \vec{i}_j = (1 - \beta)(\vec{\nabla}W)_{ij}^U \cdot \vec{i}_j + \beta(\vec{\nabla}W)_{ij}^L \cdot \vec{i}_j + \\
\xi_c \left[ ((\vec{\nabla}W)_{ij}^U \cdot \vec{i}_j) - 2(\vec{\nabla}W)_{ij}^C \cdot \vec{i}_j + (\vec{\nabla}W)_{ij}^D \cdot \vec{i}_j \right] + \\
\xi_c \left[ ((\vec{\nabla}W)_{ij} \cdot \vec{i}_j) - 2(\vec{\nabla}W)_{ij}^C \cdot \vec{i}_j + (\vec{\nabla}W)_{ij}^D \cdot \vec{i}_j \right],
\end{equation}

(2.11)

\begin{equation}
(\vec{\nabla}W)_{ji} \cdot \vec{j}_i = (1 - \beta)(\vec{\nabla}W)_{ji}^U \cdot \vec{j}_i + \beta(\vec{\nabla}W)_{ji}^L \cdot \vec{j}_i + \\
\xi_c \left[ ((\vec{\nabla}W)_{ji}^U \cdot \vec{j}_i) - 2(\vec{\nabla}W)_{ji}^C \cdot \vec{j}_i + (\vec{\nabla}W)_{ji}^D \cdot \vec{j}_i \right] + \\
\xi_c \left[ ((\vec{\nabla}W)_{ji} \cdot \vec{j}_i) - 2(\vec{\nabla}W)_{ji}^C \cdot \vec{j}_i + (\vec{\nabla}W)_{ji}^D \cdot \vec{j}_i \right],
\end{equation}

(2.12)

where \((\vec{\nabla}W)_i\) and \((\vec{\nabla}W)_j\) are the nodal gradients at the nodes \(i\) and \(j\) respectively and are calculated as the average of the gradient on the tetrahedra \(T \in C_i\), having the node \(i\) as a vertex. For example for \((\vec{\nabla}W)_i\) we can write:

\begin{equation}
(\vec{\nabla}W)_i = \frac{1}{Vol(C_i)} \sum_{T \in C_i} \frac{Vol(T)}{3} \sum_{k \in T} W_k \vec{\nabla}\Phi(i,T).
\end{equation}

(2.13)

where \(\Phi^{(i,T)}\) is the P1 finite-element basis function defined before. \((\vec{\nabla}W)_{M'} \cdot \vec{i}_j\), for the 3D case, is the gradient at the point \(M\) in Fig. 2.2 and it is computed by interpolation of the nodal gradient values at the nodes contained in the face opposite to the upwind tetrahedron \(T_{ij}\). \((\vec{\nabla}W)_{M'} \cdot \vec{j}_i\) is the gradient at the point \(M'\) in Fig. 2.2 and it is evaluated in the same way as \((\vec{\nabla}W)_{M} \cdot \vec{j}_j\).

The coefficients \(\beta, \xi_c, \xi_d\) are parameters that control the combination of fully upwind and centred slopes. The V6 scheme is obtained by choosing them to have the best accuracy on cartesian meshes, Ref.[12]:

\begin{equation*}
\beta = 1/3, \quad \xi_c = -1/30, \quad \xi_d = -2/15.
\end{equation*}

**Diffusive fluxes**

The P1 finite-element basis function, \(\phi^{(i,T)}\), restricted to the tetrahedron \(T\) is assumed to be of unit value on the node \(i\) and to vanish linearly at the remaining vertexes of \(T\). The Galerkin formulation for the diffusive terms is
obtained by multiplying the diffusive terms by $\phi^{(i,T)}$ and integrating over the domain $\Omega_h$:

$$
\int \int_{\Omega_h} \left( \frac{\partial V_j}{\partial x_j} \right) \phi^{(i,T)} \, d\Omega = \int \int_T \frac{\partial V_j}{\partial x_j} \phi^{(i,T)} \, d\Omega .
$$

Integrating by parts the right-hand side of Eq.(2.3) we obtain:

$$
\int \int_T \frac{\partial V_j}{\partial x_j} \phi^{(i,T)} \, d\Omega = \int \int_T \frac{\partial (V_j \phi^{(i,T)})}{\partial x_j} \, d\Omega - \int \int_T V_j \frac{\partial \phi^{(i,T)}}{\partial x_j} \, d\Omega =
\int_{\partial T} V_j \phi^{(i,T)} n_j \, d\sigma - \int \int_T V_j \frac{\partial \phi^{(i,T)}}{\partial x_j} \, d\Omega . \quad (2.14)
$$

In order to build the fluxes for the node $i$ consistently with the finite-volume formulation, the contribution of all the elements having $i$ as a vertex needs to be summed together as follows:

$$
\sum_{T,i \in T} \left( \int_{\partial T} V_j \phi^{(i,T)} n_j \, d\sigma - \int \int_T V_j \frac{\partial \phi^{(i,T)}}{\partial x_j} \, d\Omega \right) =
- \sum_{T,i \in T} \int \int_T V_j \frac{\partial \phi^{(i,T)}}{\partial x_j} \, d\Omega + \int_{\Gamma_h=\partial \Omega_h} \phi^{(i,T)} V_j n_j \, d\sigma . \quad (2.15)
$$

In the P1 formulation for the finite-element method, the test functions, $\phi^{(i,T)}$, are linear functions on the element $T$ and so their gradient is constant.
Moreover, in the variational formulation the unknown variables contained in $W$ are also approximated by their projection on the P1 basis function. For these reasons the integral can be evaluated directly.

### 2.4 Variational Multiscale approach for Large Eddy Simulation

In the Variational Multiscale approach for Large Eddy Simulation (VMS-LES) approach the flow variables are decomposed as follows:

\[
    w_i = \underbrace{\bar{w}_i}_{\text{LRS}} + \underbrace{w'_i}_{\text{SRS}} + \underbrace{w_{iSGS}}_{\text{SGS}}
\]  

(2.16)

where $\bar{w}_i$ are the large resolved scales (LRS), $w'_i$ are the small resolved scales (SRS) and $w_{iSGS}$ are the unresolved scales. This decomposition is obtained by variational projection in the LRS and SRS spaces respectively. In the present study, we follow the VMS approach proposed in Ref.[1] for the simulation of compressible turbulent flows through a finite volume/finite element discretization on unstructured tetrahedral grids. If $\psi_l$ are the $N$ finite-volume basis functions and $\phi_l$ the $N$ finite-element basis functions associated to the used grid, previously defined in Sec.2.3, in order to obtain the VMS flow decomposition in Eq. (2.16), the finite dimensional spaces $\mathcal{V}_{FV}$ and $\mathcal{V}_{FE}$, respectively spanned by $\psi_l$ and $\phi_l$, can be in turn decomposed as follows [1]:

\[
    \mathcal{V}_{FV} = \mathcal{V}_{FV} \bigoplus \mathcal{V}'_{FV} ; \quad \mathcal{V}_{FE} = \mathcal{V}_{FE} \bigoplus \mathcal{V}'_{FE}
\]

(2.17)

in which $\oplus$ denotes the direct sum and $\overline{\mathcal{V}}_{FV}$ and $\mathcal{V}'_{FV}$ are the finite volume spaces associated to the largest and smallest resolved scales, spanned by the basis functions $\overline{\psi}_l$ and $\psi'_l$; $\overline{\mathcal{V}}_{FE}$ and $\mathcal{V}'_{FE}$ are the finite element analogous. In Ref.[1] a projector operator $P$ in the LRS space is defined by spatial average on macro cells in the following way:
\[ \mathbf{W} = P(\mathbf{W}) = \sum_k \left( \frac{Vol(C_k)}{\sum_{j \in I_k} Vol(C_j) \sum_{j \in I_k} \psi_j} \right) \mathbf{W}_k \] (2.18)

for the convective terms, discretized by finite volumes, and:

\[ \mathbf{W} = P(\mathbf{W}) = \sum_k \left( \frac{Vol(C_k)}{\sum_{j \in I_k} Vol(C_j) \sum_{j \in I_k} \phi_j} \right) \mathbf{W}_k \] (2.19)

for the diffusive terms, discretized by finite elements. In both Eqs. (2.18) and (2.19), \( I_k = \{ j/C_j \in C_{m(k)} \} \), \( C_{m(k)} \) being the macro-cell containing the cell \( C_k \). The macro-cells are obtained by a process known as agglomeration \[13\]. The basis functions for the SRS space are clearly obtained as follows:

\[ \psi'_l = \psi_l - \overline{\psi}_l \] and \[ \phi'_l = \phi_l - \overline{\phi}_l. \]

A key feature of the VMS-LES approach is that the modeled influence of the unresolved scales on large resolved ones is set to zero, and so the SGS model is added only to the smallest resolved scales (which models the dissipative effect of the unresolved scales on small resolved ones). This leads to the following equations after semi-discretizations \[1\].

\[
\int_{C_i} \frac{\partial \rho}{\partial t} d\Omega + \int_{\partial C_i} \rho \hat{V} \cdot \mathbf{n} d\Gamma = 0 \\
\int_{C_i} \frac{\partial \rho \hat{V}}{\partial t} d\Omega + \int_{\partial C_i} \rho \hat{V} \otimes \hat{V} \cdot \mathbf{n} d\Gamma + \int_{\partial C_i} p \hat{n} d\Gamma \\
+ \frac{1}{Re} \int_{\partial \Omega} \sigma \nabla \Phi_i d\Omega + \frac{1}{Re} \int_{\Omega} \tau' \nabla \Phi_i' d\Omega = 0 \\
\int_{C_i} \frac{\partial E}{\partial t} d\Omega + \int_{\partial C_i} (E + p) \hat{V} \cdot \mathbf{n} d\Gamma + \int_{\Omega} \sigma \hat{V} \cdot \nabla \Phi_i d\Omega \\
+ \frac{\gamma}{Re Pr} \int_{\Omega} \nabla e \cdot \nabla \Phi_i d\Omega + \frac{\gamma}{Re Pr} \int_{\Omega} \mu' \nabla e' \cdot \nabla \Phi_i' d\Omega = 0 
\] (2.20)
where \( e \) denotes the internal energy (\( E = e + \frac{1}{2}V^2 \)) and \( \tau' \) is the small resolved scales SGS stress given by:

\[
\tau' = \mu' + (2S'_{ij} - \frac{2}{3}S'_{kk}\delta_{ij})
\]

with \( S'_{ij} = \frac{1}{2}(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}) \) and \( \mu' \), the small resolved scales eddy viscosity (which depends on the chosen SGS model).

One can notice that the laminar Navier-Stokes equations are recovered by substituting \( \tau' = 0 \) and \( \mu' = 0 \) in Eq. (2.20) above and that the SGS model is recovered by substituting \( \tau' = \tau, \mu' = \mu, e' = e \) and \( \Phi'_i = \Phi_i \) in the equations, where \( \tau \) and \( \mu \) denote the usual SGS stress tensor and SGS eddy viscosity, respectively.

More details about this VMS-LES methodology can be found in Ref. [1] and [4].

### 2.5 Boundary conditions

First of all, the real boundary \( \Gamma \) is approximated by a polygonal boundary \( \Gamma_h \) that can be split in two parts:

\[
\Gamma_h = \Gamma_\infty + \Gamma_b
\]

(2.21)

where the term \( \Gamma_\infty \) represents the far-fields boundary and \( \Gamma_b \) represents the body surface. The boundary conditions are set using the Steger-Warming formulation ([14]) on \( \Gamma_\infty \) and using slip or no-slip conditions on \( \Gamma_b \).

### 2.6 Time advancing

Once the equations have been discretized in space, the unknown of the problem is the solution vector at each node of the discretization as a function of time, \( \mathbf{W}_h(t) \). Consequently the spatial discretization leads to a set of ordinary differential equations in time:
\[
\frac{dW_h}{dt} + \Psi(W_h) = 0 \quad (2.22)
\]
where \( \Psi \) is the total flux, containing both convective and diffusive terms, of \( W_h \) through the \( i \)-th cell boundary divided by the volume of the cell.

**Explicit time advancing**

In the explicit case a \( N \)-step low-storage Runge-Kutta algorithm is used for the discretization of Eq. (2.22):

\[
\begin{align*}
W^{(0)}(n) &= W^{(n)}, \\
W^{(k)}(n) &= W^{(0)}(n) + \Delta t \alpha_k \Psi(W^{(k-1)}(n)), \quad k = 1, \ldots, N \\
W^{(n+1)} &= W^{(N)}(n),
\end{align*}
\]
in which the suffix \( h \) has been omitted for sake of simplicity. Different schemes can be obtained varying the number of steps, \( N \), and the coefficients \( \alpha_k \).

**Implicit time advancing**

For the implicit time advancing scheme in AERO the following second order accurate backward difference scheme is used:

\[
\alpha_{n+1} W^{(n+1)} + \alpha_n W^{(n)} + \alpha_{(n-1)} W^{(n-1)} + \Delta t^{(n)} \Psi(W^{(n+1)}(n)) = 0 \quad (2.23)
\]
where the coefficients \( \alpha_n \) can be expressed as follows:

\[
\alpha_{n+1} = \frac{1 + 2\tau}{1 + \tau}, \quad \alpha_n = -1 - \tau, \quad \alpha_{n-1} = \frac{\tau^2}{1 + \tau} \quad (2.24)
\]
where \( \Delta t^{(n)} \) is the time step used at the \( n \)-th time iteration and

\[
\tau = \frac{\Delta t^{(n)}}{\Delta t^{(n-1)}}, \quad (2.25)
\]
the nonlinear system obtained can be linearised as follows:
\[ \alpha_{n+1} W^{(n)} + \alpha_n W^{(n)} + \alpha_{(n-1)} W^{(n-1)} + \Delta t^{(n)} \Psi(W^{(n)}) = - \left[ \alpha_{n+1} + \delta t^{(n)} \frac{\partial \Psi}{\partial W}(W^{(n)}) \right] (W^{(n+1)} - W^{(n)}) \]  

(2.26)

Following the deflect-correction approach, the jacobians are evaluated using the 1st order flux scheme (for the convective part), while the explicit fluxes are composed with 2nd order accuracy. The resulting linear system is solved by a Schwarz method.