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Twisted Whitney towers and concordance of links

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Contents

Introduction

In this thesis we describe *how the study of links is affected by allowing an extra dimension*.

The study of smooth *knots* $S^1 \hookrightarrow S^3$ and *links* $S^1 \sqcup \cdots \sqcup S^1 \hookrightarrow S^3$ up to isotopy is a central problem in the area of low-dimensional topology, i.e. the study of manifolds of dimension ≤ 4. Every closed oriented 3-manifold is obtained by taking a link in S^3, removing a tubular neighborhood of it and gluing these solid tori back in the sphere along a diffeomorphism of the boundary. This procedure is called *Dehn surgery* and different link diagrams describe diffeomorphic 3-manifold via surgery if and only if they are related by a set of combinatorial transformations known as *Kirby moves*. Surgery diagrams can also be interpreted as gluing instructions of 4-dimensional handles to $S^3 = \partial D^4$. The theory of Kirby calculus uses this idea to give a combinatorial description of 4-manifolds, an accessible account of this is [**GS99**].

Given an isotopy of knots $\phi : S^1 \times [0,1] \to S^3$ there is an embedding $\widehat{\phi} : S^1 \times [0,1] \hookrightarrow S^3 \times [0,1]$ given by $\widehat{\phi}(x,t) = (\phi(x,t),t)$ of an annulus whose boundary components are the original knots ϕ_0 and ϕ_1, respectively in the spheres $S^3 \times \{0\}$ and $S^3 \times \{1\}$. The case of links is analogous and gives disjointly embedded annuli. Such a family of annuli is called a *concordance* and the links are called concordant.

Fox and Milnor observed in 1966 [**FM66**] that knots with the operation of connected sum form a group modulo concordance. The structure of this group is still very misterious; Livingston wrote a survey of the state of the art in 2004 [**Liv04**]. Invariants have been developed to distinguish knots and links up to concordance, but the story is far from its end. For example a modification of links called mutation is known to be able to change the concordance class of a link but is not detected by the signature [**KL01**], one of the standard concordance invariants of links.

In detail, the goal of this work is to give an account of the theory of Whitney towers as a way to study singular concordances of links, i.e. generically immersed annuli. This framework has been developed in the last fifteen years by Conant, Schneiderman and Teichner, see [**CST11**] for a survey.

In Chapter 1 we explain the theory in its simplest, framed form and show its connection with a class of link invariants defined by Milnor in his early works [**Mil54, Mil57**] and with the problem of deciding whether a link is slice (i.e. concordant to unlink) or not.

In Chapter 2 we address some subtler problem regarding twisting in Whitney towers and the corresponding obstruction theory. Twisted towers are used to define a notion of twisted concordance and a corresponding filtration on the set of links. The structure of this filtration is not fully understood yet and it seems to suggest the existence of a new class of concordance invariants generalizing the classical Arf invariant of knots.

In Chapter 3 we explain a general procedure to build string link invariants from ribbon Hopf algebras and a recent result of Meilhan and Suzuki [**MS14**] describing concordance information contained in the invariant associated to the quantized h-adic universal enveloping algebra $U_h(\mathfrak{sl}_2)$.

As future project, we aim at finding concordance information in quantum invariants coming from other ribbon Hopf algebras and to relate it to the obstruction theory of twisted Whitney towers.

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CHAPTER 1

An obstruction theory for slice disks

1.1 Slice links

A link $L \subset S^3$ is the image of a smooth embedding $\sqcup S^1 \hookrightarrow S^3$, its connected components are knots and we assume they are finite and write $L = L_1 \cup \cdots \cup L_m$. The intrinsic topology of a link is obvious, but we are interested in distinguishing them as embeddings up to ambient isotopies of S^3, i.e. we consider $L, L' \subset S^3$ equal if there is a smooth map $H : S^3 \times [0,1] \to S^3$ such that $H(\cdot, 0) = \mathrm{id}_{S^3}$, $H(\cdot, 1)(L) = L'$ and $H(\cdot, t)$ is a diffeomorphism of S^3 for every t. A link is trivial if its embedding extends to a smooth embedding $\sqcup D^2 \hookrightarrow S^3$.

One of the classical goals of knot theory is to decide whether a link is trivial or not. The main concern of this chapter is to study links in a 4-dimensional perspective: thinking $S^3 = \partial D^4$ we call a link slice if its embedding extends to a smooth embedding $\sqcup D^2 \hookrightarrow D^4$. There are infinitely many nontrivial slice links, see for example Figure 1.1.

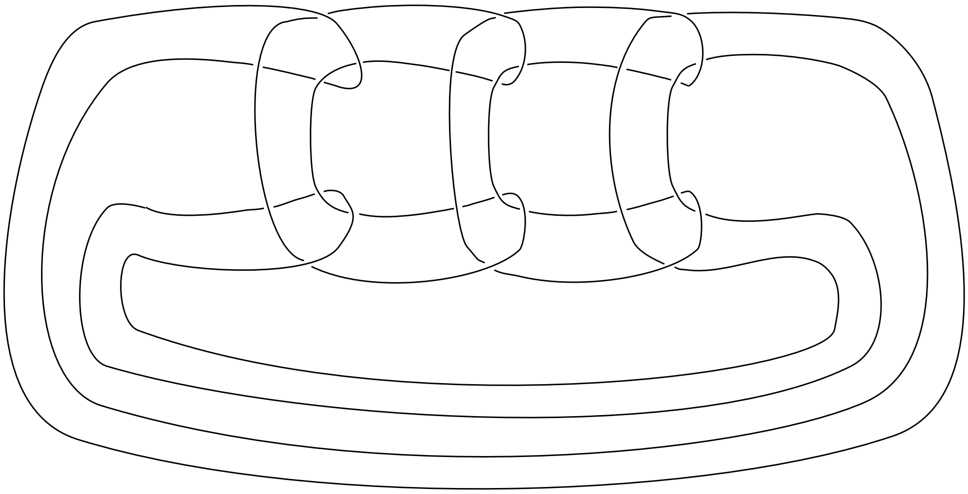

Figure 1.1: Rubber band link

Not all links are slice and we develop below obstructions to prove this. By the way, one can always assume to be in the following situation

PROPOSITION 1. *Given $L \subset S^3$, each component is the boundary of a compact connected orientable surface properly embedded in D^4 and these can be chosen in such a way that each* *pair intersects in finitely many interior points. The same statement is also true with generically immersed disks instead of properly embedded surfaces.*

Proof. The link L is given by an embedding $\phi : \sqcup S^1 \hookrightarrow S^3$, we denote ϕ_i its restrictions to the components of L. By Seifert's algorithm [**Kau87**] each one admits an extension to a smooth embedding $\widehat{\phi}_i : \Sigma_i \hookrightarrow S^3$ where Σ_i is a compact connected orientable surface with $\partial\Sigma_i = S^1$. Now we push interiors inside D^4, this means we trade $\widehat{\phi}_i$ with a proper embedding $\widetilde{\phi}_i : \Sigma_i \hookrightarrow D^4$ as follows. Choose a smooth function $\lambda_i : \Sigma_i \to \mathbb{R}$ with value 1 on the boundary circle, with $\epsilon \leq \lambda_i < 1$ on the interior and such that $\|\nabla\lambda_i\|_\infty \leq \epsilon$ and take $\widetilde{\phi}_i = \lambda_i\widehat{\phi}_i$. The latter condition guarantees that $\| \mathrm{d}\,\widetilde{\phi} - \mathrm{d}\,\widehat{\phi}_i\| \leq C\epsilon$ for some costant C and thus $\widetilde{\phi}_i$ is an immersion because $\widehat{\phi}_i$ is, furthermore $\widetilde{\phi}_i$ is injective and thus an embedding. By perturbing these embedded surfaces in D^4 they intersect in a compact 0-dimensional manifold. For the second part of the statement, the fact that D^4 is simply connected guarantees that ϕ_i extends to a continuous map $\widehat{\phi}_i : D^2 \to D^4$ and by standard approximation theorems in differential topology this is homotopic relative to the boundary to a generic proper immersion, i.e. a smooth map that is an embedding on the boundary and an immersion on the interior with isolated double points. □

We will assume from now on that all surfaces or immersed disks bounding an oriented link are respectively properly embedded and generically immersed in D^4 and have orientation induced by the link.

1.2 4d interpretation of linking number

A first obstruction to sliceness is given by linking numbers of pairs of components in a link. The linking number of $L = L_1 \cup L_2$ oriented link is defined by noticing that $H_1(S^3 \setminus L_1; \mathbb{Z}) \cong \mathbb{Z}$ and a generator is given by $[\mu_1]$ meridian of L_1, then

$$[L_2] = \mathrm{lk}(L_1, L_2)[\mu_1] \in H_1(S^3 \setminus L_1; \mathbb{Z})$$

See [**Kau87**] for more about this and a simple recipe to compute this number from a planar diagram of L.

PROPOSITION 2. *Given $L = L_1 \cup L_2$ oriented link, any two surfaces bounding L in D^4 have intersection number $\mathrm{lk}(L_1, L_2)$.*

Proof. Let $F_1, F_2 \subset D^4$ be two surfaces bounding respectively L_1, L_2. In general position these surfaces intersect in finitely many interior points $p_1, \dots, p_k$. Choose $D_1, \dots, D_k \subset F_2$ disjoint disks around each intersection point and put $\widetilde{F_2} = F_2 \setminus (\cup_i \mathrm{int}(D_i))$, this is a surface with $\partial\widetilde{F_2} = L_2 \sqcup (\sqcup_i \partial D_i)$ and it inherits an orientation from F_2.

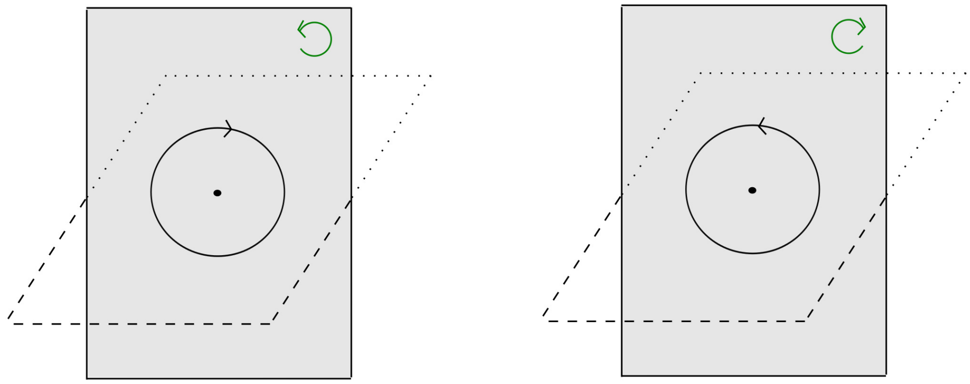

Figure 1.2: Generic intersections

Up to shrinking the disks, around each p_i we have a local picture as in Figure 1.2 and ∂D_i is thus a meridian of F_1, i.e. the boundary of the fiber disk over p_i in the tubular neighborhood $N(F_1) \subset D^4$. Then $\widetilde{F_2}$ gives an homology in $D^4 \setminus F_1$ between K_2 and a union of meridians of F_1 and

$$[L_2] = [\partial D_1] + \cdots + [\partial D_k] \quad \in H_1(D^4 \setminus F_1; \mathbb{Z})$$

Now $N(F_1) \restriction_{L_1} = N(L_1)$ tubular neighborhood of $L_1 \subset S^3$ and being F_1 an oriented surface with boundary $N(F_1)$ is a trivial disk bundle, therefore every meridian ∂D_i of F_1 is homologous in $N(F_1)$ to a fixed meridian μ_1 of $L_1 \subset S^3$ with an orientation governed by a sign $\mathrm{sgn}(p_i)$ that says if F_1 intersects F_2 in p_i positively or negatively, thus

$$[L_2] = \sum_{i=1}^{k} [\partial D_i] = \left(\sum_{i=1}^{k} \mathrm{sgn}(p_i) \right) [\mu] = (\# F_1 \cap F_2)[\mu] \quad \in H_1(D^4 \setminus F_1; \mathbb{Z})$$

where $\# F_1 \cap F_2$ is the intersection number. The proof ends by noticing that using Mayer-Vietoris on the decomposition $D^4 = N(F_1) \cup \overline{D^4 \setminus N(F_1)}$ we get an exact sequence

$$H_2(D^4) \to H_1(N(F_1) \cap \overline{D^4 \setminus N(F_1)}) \to H_1(N(F_1)) \oplus H_1(\overline{D^4 \setminus N(F_1)}) \to H_1(D^4)$$

Being D^4 contractible the middle arrow is an isomorphism and we also know that $N(F_1) \cong F_1 \times D^2$, $N(F_1) \cap \overline{D^4 \setminus N(F_1)} \cong F_1 \times S^1$ and $\overline{D^4 \setminus N(F_1)} \simeq D^4 \setminus F_1$. Using then Künneth we finally get that the inclusion $S^3 \setminus L_1 \hookrightarrow D^4 \setminus F_1$ induces an iso $H_1(S^3 \setminus K_1) \cong H_1(D^4 \setminus F_1)$ and this gives

$$\mathrm{lk}(L_1, L_2)[\mu_1] = [L_2] = (\# F_1 \cap F_2)[\mu_1] \quad \in H_1(S^3 \setminus L_1; \mathbb{Z})$$

We point out that this also gives a proof of $\mathrm{lk}(L_1, L_2) = \mathrm{lk}(L_2, L_1)$, not obvious from the definition of linking number. $\square$

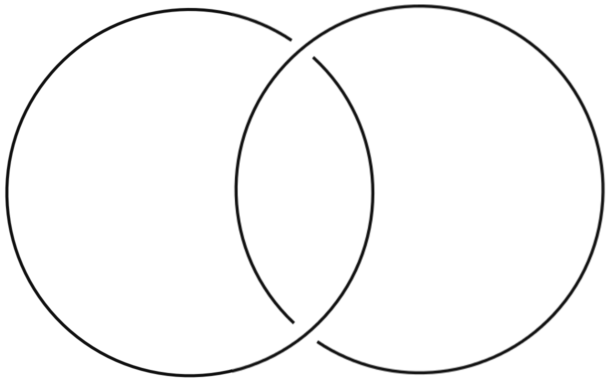

Figure 1.3: Hopf link

A simple consequence of this is that the Hopf link in Figure 1.3 is not slice, in fact one can choose arbitrary orientations on the components L_i and in any case $\mathrm{lk}(L_1, L_2) = \pm 1$, therefore L cannot bound disjoint disks in D^4.

1.3 Bing doubling

Given an oriented knot K, a longitude is any embedded circle in the boundary of a tubular neighborhood $\partial N(K)$ that is homologous to K in $N(K)$ and nullhomologous in $S^3 \setminus \mathrm{int}(N(K))$. Such a longitude is unique up to ambient isotopy in $\partial N(K)$ and can be obtained as generic intersection of a Seifert surface Σ of K with $\partial N(K)$. A closed collar of $K \subset \Sigma$ gives an embedded strip $S \subset N(K)$ and it is orientable because Σ is. The orientation of K induces an orientation on S and the bounding longitude. Figure 1.4 shows how to crop an oriented square Q out of S, rotate it of $\pi/2$ in the direction of a positive meridian and expand it to intersect $S \setminus Q$ in two clasps.

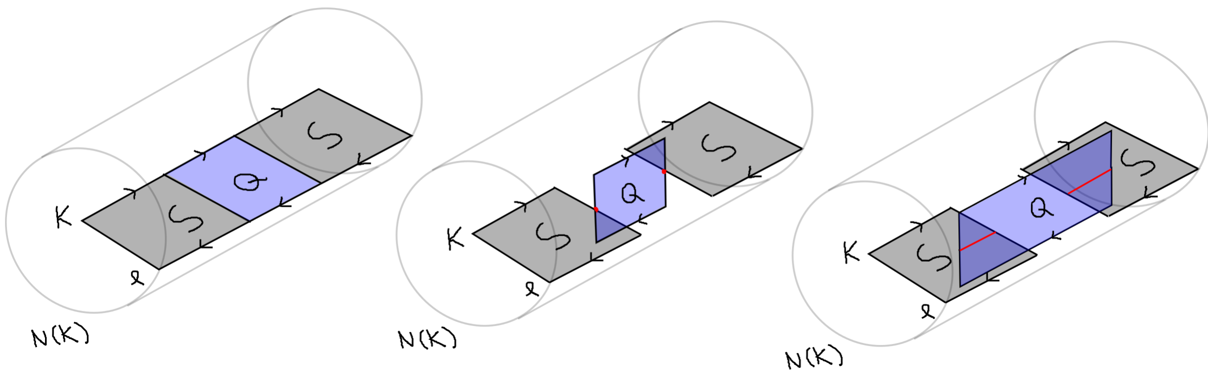

Figure 1.4: Bing doubling

We call the oriented link $BD(K) = \partial(S \setminus Q) \cup \partial Q$ Bing double of K. By construction, it has two unknotted componets whose linking number is 0. If $L = L_1 \cup \cdots \cup L_m$ we call $BD_i(L)$ the (m+1)-link obtained by Bing doubling the i-th component of L.

We are mainly interested in Bing doubling as a way to kill linking numbers between components of a given link, as illustrated in Figure 1.5 where a Hopf link is Bing doubled to get the so called Borromean rings, whose pairwise linking numbers are all 0.

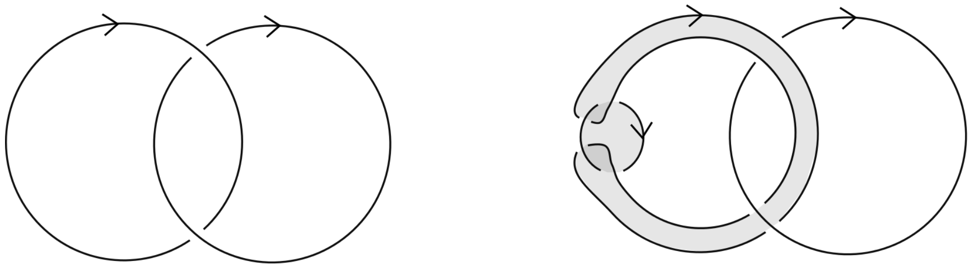

Figure 1.5: Bing double of Hopf link

By construction, after a Bing doubling on a component the newly created components have two natural bounding disks coming from a collar and living inside a tubular neighborhood of the old component, therefore both have linking number 0 with any other component of the link.

PROPOSITION 3. *If $L = L_1 \cup \cdots \cup L_m$ is an oriented slice link, then $BD_i(L)$ is slice for every i*

Proof. We prove the claim for $L = K$ slice knot, the case of a slice link is analogous. See Figure 1.6 for an example. Let $BD(K) = K_1 \cup K_2$, by definition the two components are both unknot and come with two canonical bounding disks Q and $S \setminus Q$ as described in the definition of Bing double, both living in a tubular neighborhood $N(K)$ of $K \subset S^3$ and intersecting in two clasps. We can build two disks D_1, D_2 bounding the components, properly embedded in D^4 and intersecting in exactly two points in the following way.

Extend $K_1 \cup K_2$ radially inside D^4 by ϵ, this means extending the initial embedding $\phi : S^1 \sqcup S^1 \hookrightarrow S^3$ to a proper embedding $\widehat{\phi} : (S^1 \sqcup S^1) \times [0,1] \hookrightarrow D^4$ given by $\widehat{\phi}(x,t) = (1-t)\phi(x) + t(1-\epsilon)\phi(x)$. Now each $K_i \subset S^3$ bounds an annulus $A^i_{[1,1-\epsilon]}$ whose second boundary component is $(1-\epsilon)K_i \subset (1-\epsilon)S^3$, cap off $A^1_{[1,1-\epsilon]}$ with the disk $(1-\epsilon)Q \subset (1-\epsilon)S^3$ and further extend $(1-\epsilon)K_2$ radially by ϵ. This gives another annulus $A^2_{[1-\epsilon,1-2\epsilon]}$ and finally cap off this with the disk $(1-2\epsilon)(S \setminus Q) \subset (1-2\epsilon)S^3$. The required disks are given by $D_1 = A^1_{[1,1-\epsilon]} \cup (1-\epsilon)Q$ and $D_2 = A^2_{[1,1-\epsilon]} \cup A^2_{[1-\epsilon,1-2\epsilon]} \cup (1-2\epsilon)(S \setminus Q)$ and they intersect in two points $p, q \in (1-\epsilon)S^3$.

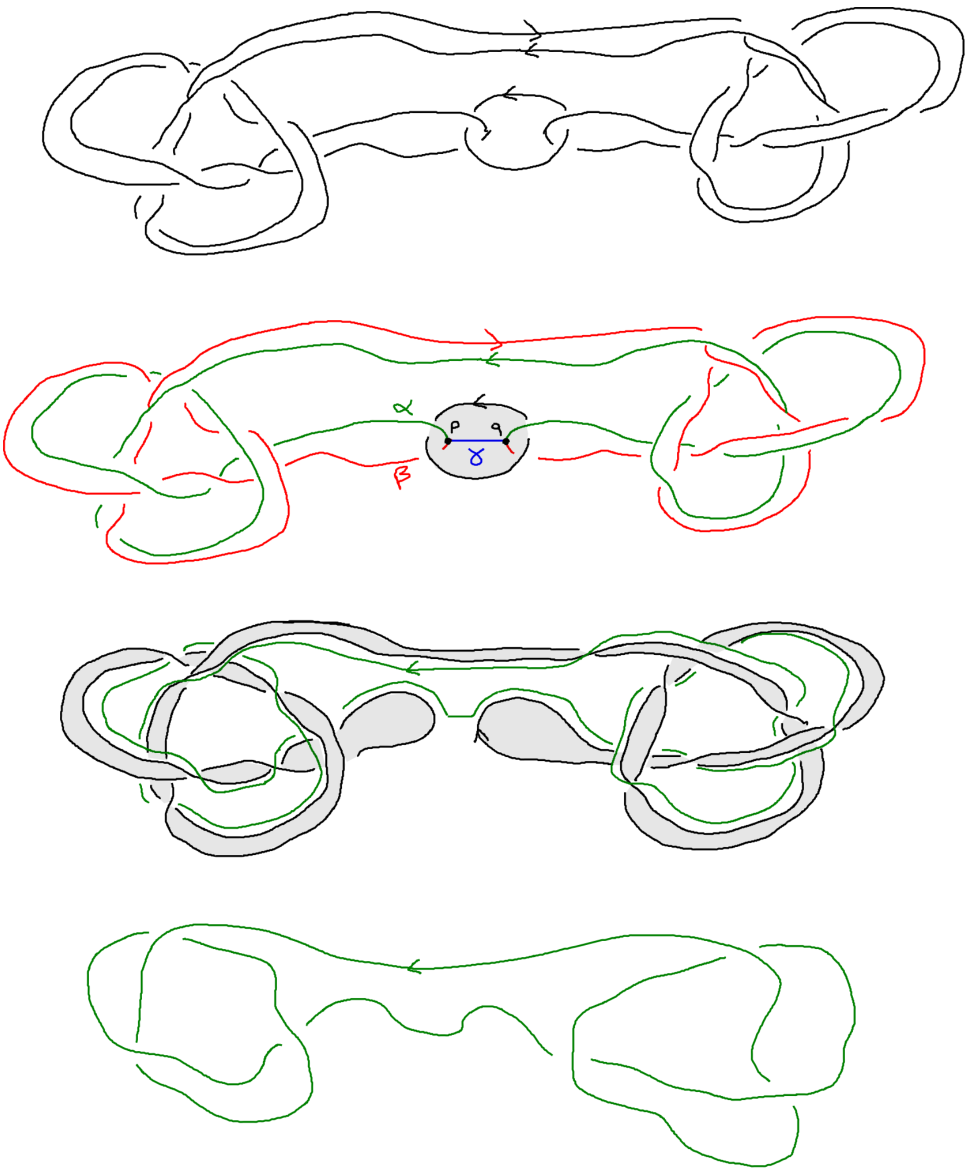

Figure 1.6: Slice disks for the Bing double of a square knot

Finally we remove these intersection points by means of a Whitney move (see [**Sco05**] for a description of this classical trick of 4-dimensional topology). The intersection points p, q divide $(1-\epsilon)K_2$ in two arcs, call α (green in Figure 1.6) the arc that does not intersect the longitude of $(1-\epsilon)K$ and β (red in Figure 1.6) the other, they are oriented as in the definition of Bing double. Take a path $\gamma \subset (1-\epsilon)Q$, by choosing a compatible orientation this gives an oriented closed curve $\alpha \cup \gamma$ isotopic to $(1-\epsilon)K \subset (1-\epsilon)S^3$.

Build a Whitney disk as follows. Extend $\alpha \cup \gamma$ radially by ϵ inside D^4 and normally in the opposite direction of a longitude of $\alpha \cup \gamma \subset (1-\epsilon)S^3$. Now $\alpha \cup \gamma$ bounds an annulus $A^W_{[1-\epsilon, 1-2\epsilon]}$, by further extending this annulus radially inside D^4 by ϵ and then capping off with a slice disk for K we get the required Whitney disk W. By construction $W \cap D_1 = \gamma$, $W \cap D_2 = \alpha$ and the disk has framing $\mathrm{lk}(\alpha \cup \gamma, \beta \cup \gamma) = 0$. $\square$

The converse problem of deciding whether a knot or link whose Bing double is slice is or not itself slice is open, see [**Cim06**] for an account of what is known and a partial result.

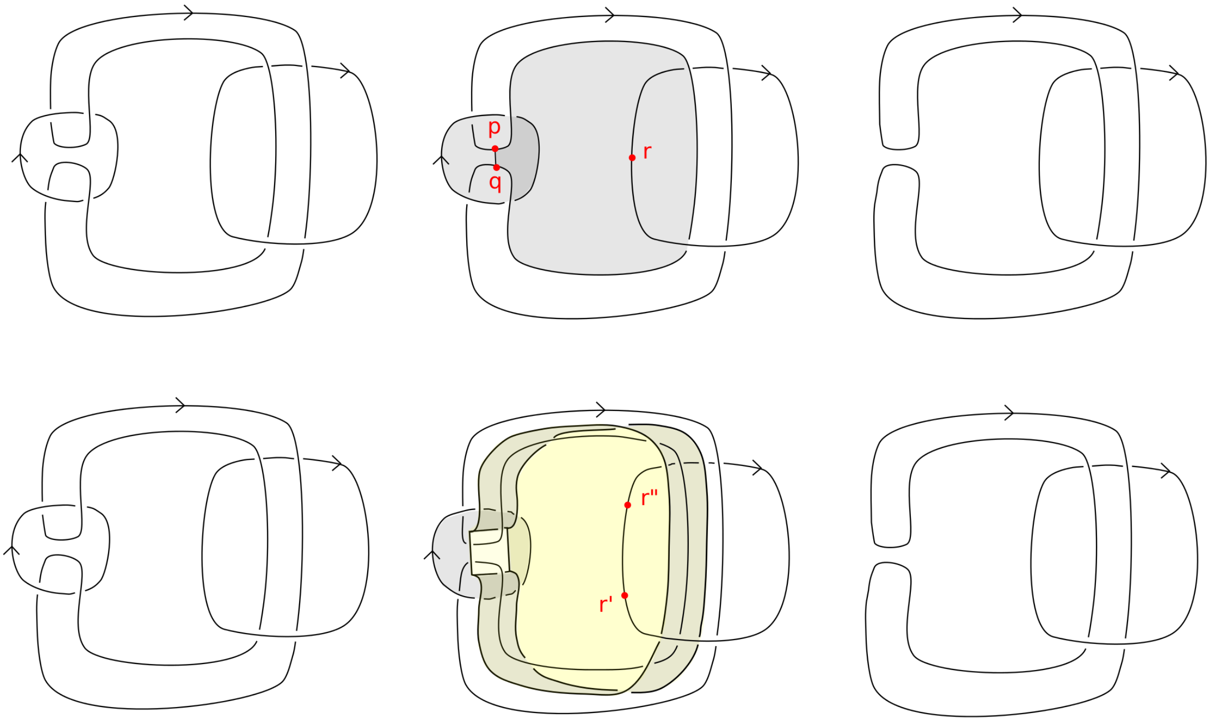

Figure 1.7: Failure of a Whitney move

If we start with an oriented link L that is not slice, a Bing doubling can (and will, if the previous open problem turns out to have answer yes) be nonslice. As an example, consider again the Borromean rings obtained by Bing doubling the Hopf link. Figure 1.7 shows three bounding disks and an attempt to make them disjoint by means of a Whitney move as in the previous proof. Here the move fails because the green Whitney disk W paring the intersections $D_1 \cap D_2 = \{p, q\}$ also intersects D_3 in a point r and after the move that removes p, q two new intersection points r', r'' are created.

1.4 Whitney towers and intersection trees

The failure of a Whitney move could be due to a bad initial choice of disks D_i or Whitney disk W but L may still be slice. We introduce now a framework due to Conant, Schneiderman and Teichner that gives obstructions to the existence of such a working family of disks. In particular this will give new obstructions for sliceness. We refer to [**CST14**] for the most recent description of this framework.

DEFINITION 1. *An oriented link $L \subset S^3$ bounds an order n Whitney tower $\mathcal{W} = \cup_k \mathcal{W}^k \subset D^4$ if the following conditions are satisfied:*

- *each component L_i bounds a generically immersed disk $W_i \subset D^4$ with compatible orientation, these are the order 0 disks of $\mathcal{W}$ and we put $\mathcal{W}^0 = \cup W_i$, we call the* *transverse intersection points in $\mathcal{W}^0$ order 0 intersections points, they come with a sign obtained by comparing with the ambient standard orientation of D^4*

- *for $0 < k \leq n$ the set of order $k-1$ intersection points has a partition in pairs (p^+, p^-) with $p^+, p^- \in W_I \cap W_J$ positive and negative point of intersection between disks of (possibly different) order $\leq k-1$, for each pair there is an immersed Whitney disk $W_{(p^+,p^-)} \subset D^4$ that pairs the intersections, it has an arbitrary orientation, these are the order k disks of $\mathcal{W}$ and we put $\mathcal{W}^k = \cup W_{(p^+,p^-)} \cup \mathcal{W}^{k-1}$, we call the transverse intersection points in $\mathcal{W}^k \setminus \mathcal{W}^{k-1}$ order k intersection points, they come with a sign obtained by comparing with the ambient standard orientation of D^4*
- *the boundaries of all disks are disjointly embedded and their interiors are in generic position*
- *all disks are framed (see Chapter 2 for framings and twisted Whitney towers)*

To summarize who intersects who, we associate to a Whitney tower $\mathcal{W}$ a combinatorial forest $t(\mathcal{W})$ using the following algorithm. To each order 0 disk W_i associate the rooted tree consisting of an edge with one vertex labeled by i. Recursively, to each order $k > 0$ disk pairing two intersection points between disks of lower order with rooted trees I and J associate the rooted product (I, J), i.e. the rooted tree obtained identifying the roots of I and J and adding a rooted leg to the gluing point in such a way that a clockwise walk around it and starting from the new leg first meets first I and then J. To an intersection point $p \in W_I \cap W_J$ we associate the unrooted tree $t_p = \langle I, J\rangle$, i.e. the same as (I, J) but with no new rooted leg added. Then $t(\mathcal{W}) = \sqcup t_p$ where the union runs over all unpaired intersection points p.

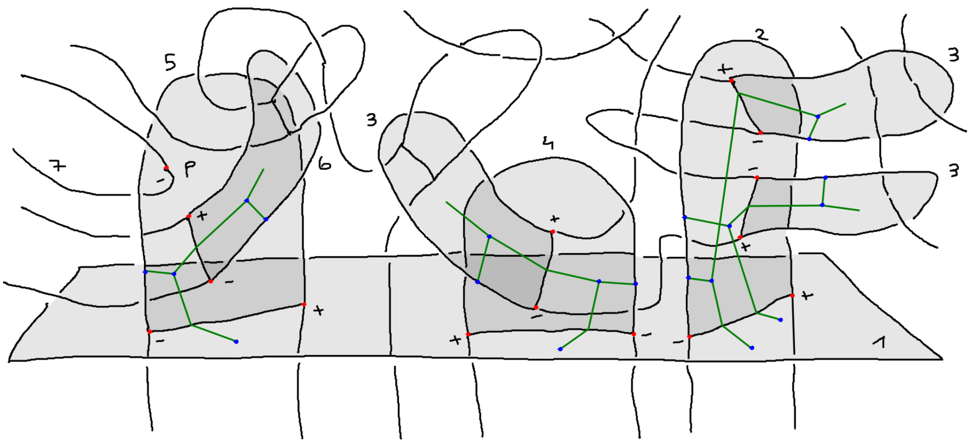

Figure 1.8: Intersection forest of a Whitney tower

All trees have vertices of degree one or three and the latter are at least n for each tree. By construction, vertices of degree one except the root come with a label in $\{1, \ldots, m\}$. Each rooted tree can be thought embedded in $\mathcal{W}$ with each trivalent vertex and rooted edge in the interior of a disk and each edge not containing the root as a path crossing a Whitney arc exactly once as in Figure 1.8. By convention, we choose the embedding in such a way that descending a tree from the root to the leaves the two edges spreading out of each trivalent vertex enclose the negative intersection point of the corresponding Whitney disk. In this way each trivalent vertex inherits a cyclic orientation from the orentation of a Whitney disk. The unrooted trees have the same trivalent oriented vertices of the rooted ones.

At the price of increasing the number of disks and intersection points, we can always arrange a Whitney tower to be split by means of finger moves as in Figure 1.9. This means that each disk of positive order has either exactly one intersection point, or exactly two intersection points paired by another disk or no intersection points at all. If $\mathcal{W}_S$ is such a tower, its disks of positive order can be partitioned in subtowers $\mathcal{W}_p$ with exatly one unpaired intersection point p each one. Clearly $t(\mathcal{W}) = t(\mathcal{W}_S)$ and the forest can be embedded in $\mathcal{W}_S$ (it can only be embedded tree by tree in $\mathcal{W}$).

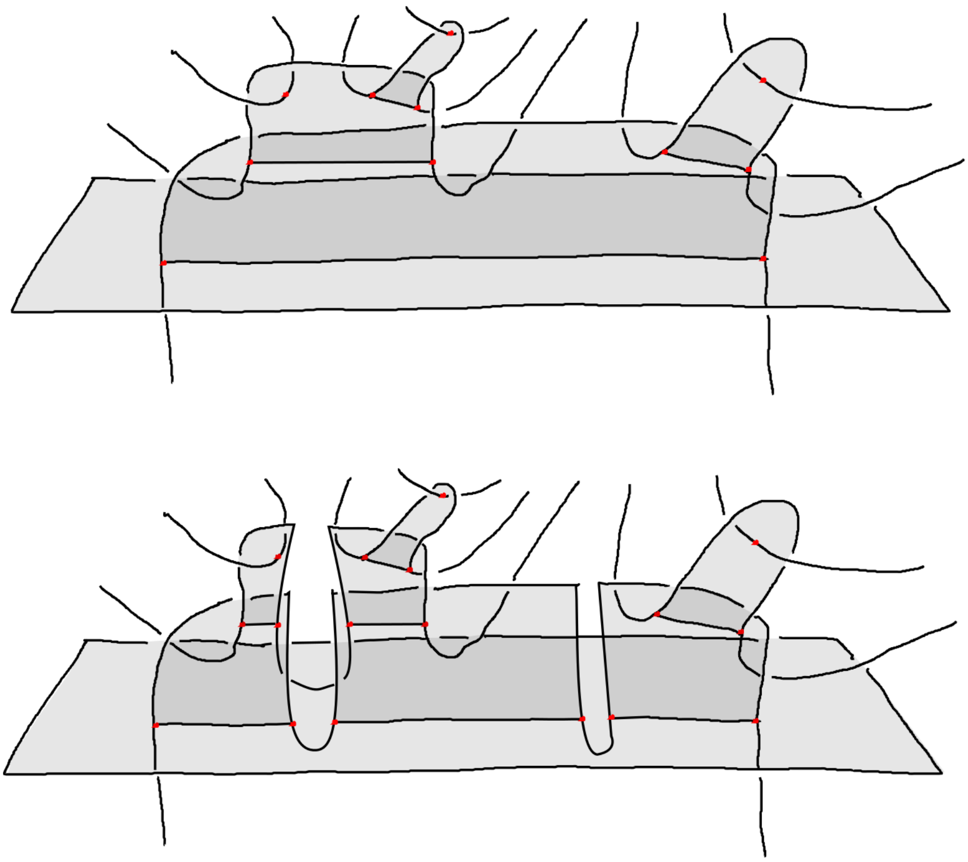

Figure 1.9: Split towers

Fixed m, we call $\mathcal{T}(m)$ the free abelian group generated by finite oriented unrooted trees with vertices of degree one or three and univalent vertices labeled on $\{1, \ldots, m\}$, modulo label preserving isomorphisms and the two relations in Figure 1.10. We denote $\mathcal{T}_n$ the subgroup generated by trees with n vertices of degree 3.

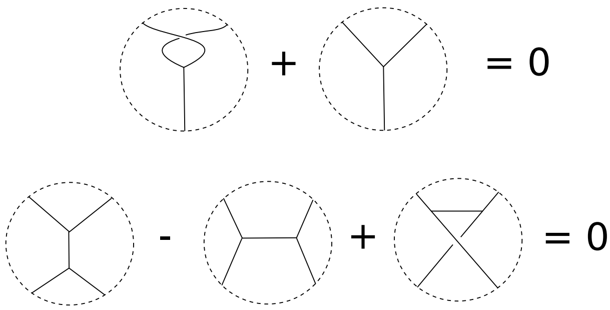

Figure 1.10: AS and IHX relations

DEFINITION 2. *Given a Whitney tower $\mathcal{W}$ of order n on a link with m components, its intersection invariant is defined as*

$$\tau_n(\mathcal{W}) = \sum \epsilon_p t_p \in \mathcal{T}_n$$

The sum is taken over all unpaired intersection points of order n and ϵ_p is the sign of the intersection point p whose associated unrooted tree is t_p as defined above.

Thanks to AS relation $\tau_n(\mathcal{W})$ does not depend on the arbitrarily chosen orientations of Whitney disks of positive order, the IHX has an important role in the following

THEOREM 1. *Given $L \subset S^3$ oriented link and $\mathcal{W}$ order n Whitney tower on it then $\tau_n(\mathcal{W}) = 0$ if and only if L bounds an order $n+1$ Whitney tower.*

see Theorem 2 in [**ST04**] for a proof.

These obstructions refer to particular tower, but we explain in Section 1.7 how it is possibile to connect them to data purely depending on the link. This connection will show, for example, that the Borromean link is not slice because the Whitney tower of order 1 described in Figure 1.7 has intersection invariant

$$\tau_1(\mathcal{W}) = 1 -\!\!\!<^{2}_{3} \neq 0 \in \mathcal{T}_1$$

However, we emphasize that if we relax our requests on the genus of surfaces bounding a link $L \subset S^4$ in D^4, it turns out that it bounds disjoint properly embedded connected surfaces if and only if the linking numbers vanish. One direction is due to Proposition 2, the other is more subtle and depends on the fact that L bounds a Whitney tower of order 1 thanks to Theorem 1 and this can be converted to a class 2 grope whose bottom stages give the required surfaces as described in Proposition 4 later.

1.5 Realization of arbitrary obstructions

The goal of this section is to show that any element in $\mathcal{T}_n$ occurs as intersection invariant $\tau_n(\mathcal{W})$ of an order n Whitney tower bounding a link with m components. What follows is a brief account of Section 3 in [**CST12**] and it will clarify the role of Bing doubling as a general way to raise the order of an intersection obstruction as already done before with the Hopf and Borromean link.

THEOREM 2. *Fixed m, for each $t \in \mathcal{T}_n$ on labels $\{1, \ldots, m\}$ there exists an oriented link $L \subset S^3$ with m components and a Whitney tower $\mathcal{W}$ of order n bounding L such that $\tau_n(\mathcal{W}) = t$.*

Proof. Assume first that t has distinct labels on vertices of degree one. We proceed by induction on n number of vertices of degree three. Assume $n = 0$ and $t = ke$ with $k \in \mathbb{Z}$ and e a single edge with distinct labels $i, j \in \{1, \ldots, m\}$ on vertices.

For $k = \pm 1$ this intersection tree can be realized by taking a split link $L = U^{(m-2)} \sqcup H$ where $U^{(m-2)}$ is the unlink with $m-2$ components and any label except i and j, H is instead a Hopf link whose componets have labels i and j. A suitable Whitney tower of order 0 $\mathcal{W}$ on L with $\tau_0(\mathcal{W}) = t$ is then given by taking disjoint disks bounding $U^{(m-2)}$ and two more disks W_i and W_j bounding H disjoint from the previous ones and intersecting each other in one point, the sign can be adjusted by choosing appropriate orientations on H.

The case $|k| > 1$ can be handled by taking $|k| - 1$ copies of H with same choices of orientations and labelings, then choosing $|k| - 1$ bands in S^3 connecting the i components and $|k| - 1$ bands connecting the j components. This gives a new link with m components with a natural Whitney tower of order 0 and $|k|$ unpaired intersection points of order 0 with same sign, the tower is obtained by performing boundary band sums of disks of order 0 of the individual towers along the chosen bands, see Figure 1.11 for the case $m = k = 2$. Any sum of trees of the previous type can be realized by performing again band sums of individual links realizing each one, with corresponding boundary band sums of 0-disks in their Whitney towers.

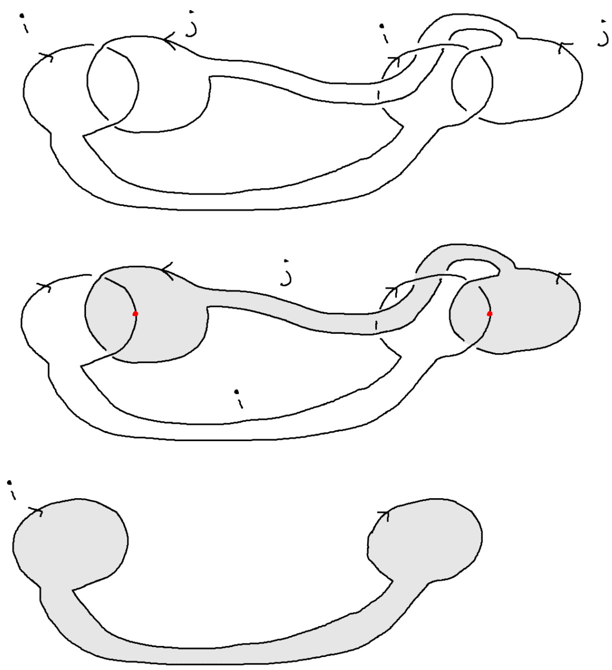

Figure 1.11: Band sum

Assume now that t has $n > 0$ vertices of degree three. If t is a single tree, a choice of a root $r \in t$ induces a partial order on the set $\mathcal{V}(t)$ of vertices of t: say $v_1 \leq v_2$ if there exists a path from r to v_2 that contains v_1. Call leaves $\mathcal{L}(t)$ (with respect to r) the set of maximal elements in $\mathcal{V}(t)$ and preleaves the maximal elements in $\mathcal{V}(t) \setminus \mathcal{L}(t)$ (this is nonempty because $n > 0$). Any preleaf has exactly two adjacent leaves because t has only vertices of degree 1 or 3.

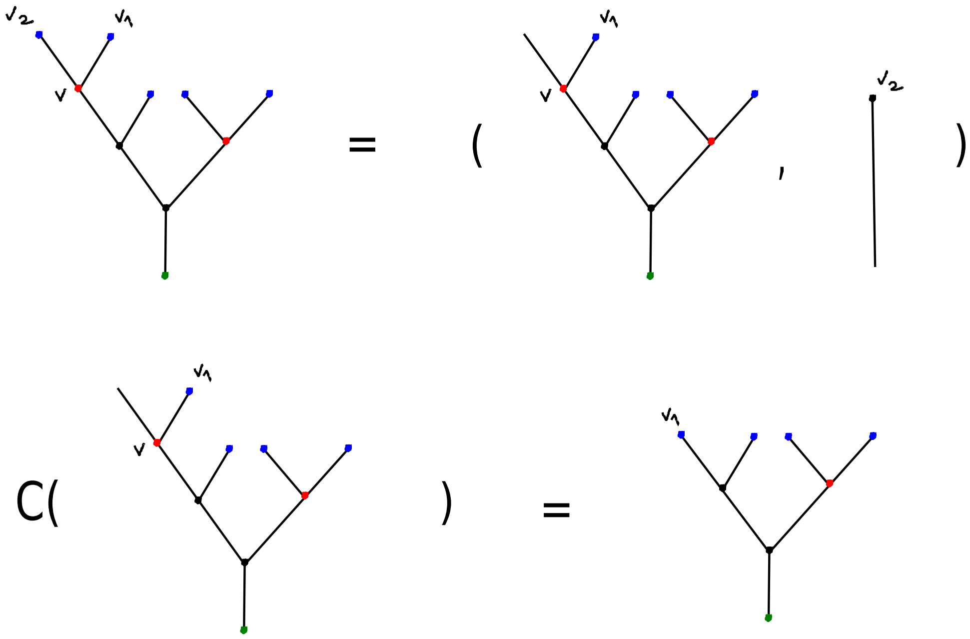

Figure 1.12: Undoubling

Fix now an arbitrary root $r \in t$ and any preleaf v with respect to r, call v_1, v_2 its adjacent leaves and assume $v_1 < v_2$ according to the cyclic ordering around v given by hypothesis. Then $t = (t', e)$ where t' has n vertices of degree three and a root obtained by deleting the label $l(v_2)$, e is an edge with a root and a vertex labelled $l(v_2)$. By induction one can build a link $L' \subset S^3$ with $m-1$ components and a Whitney tower $\mathcal{W}'$ of degree $n-1$ bounding L' such that $\tau_{n-1}(\mathcal{W}') = c(t')$ unrooted three with $n-1$ vertices of degree three obtained by collapsing the root of t' and forgetting v. Taking $L = BD_{l(v_1)}(L')$ we get a link with m components and if we agree to label $l(v_2)$ the unknotted component obtained via rotation of $\pi/2$ in the definition of Bing double we claim that L bounds a Whitney tower $\mathcal{W}$ of order n such that $\tau_n(\mathcal{W}) = t$.

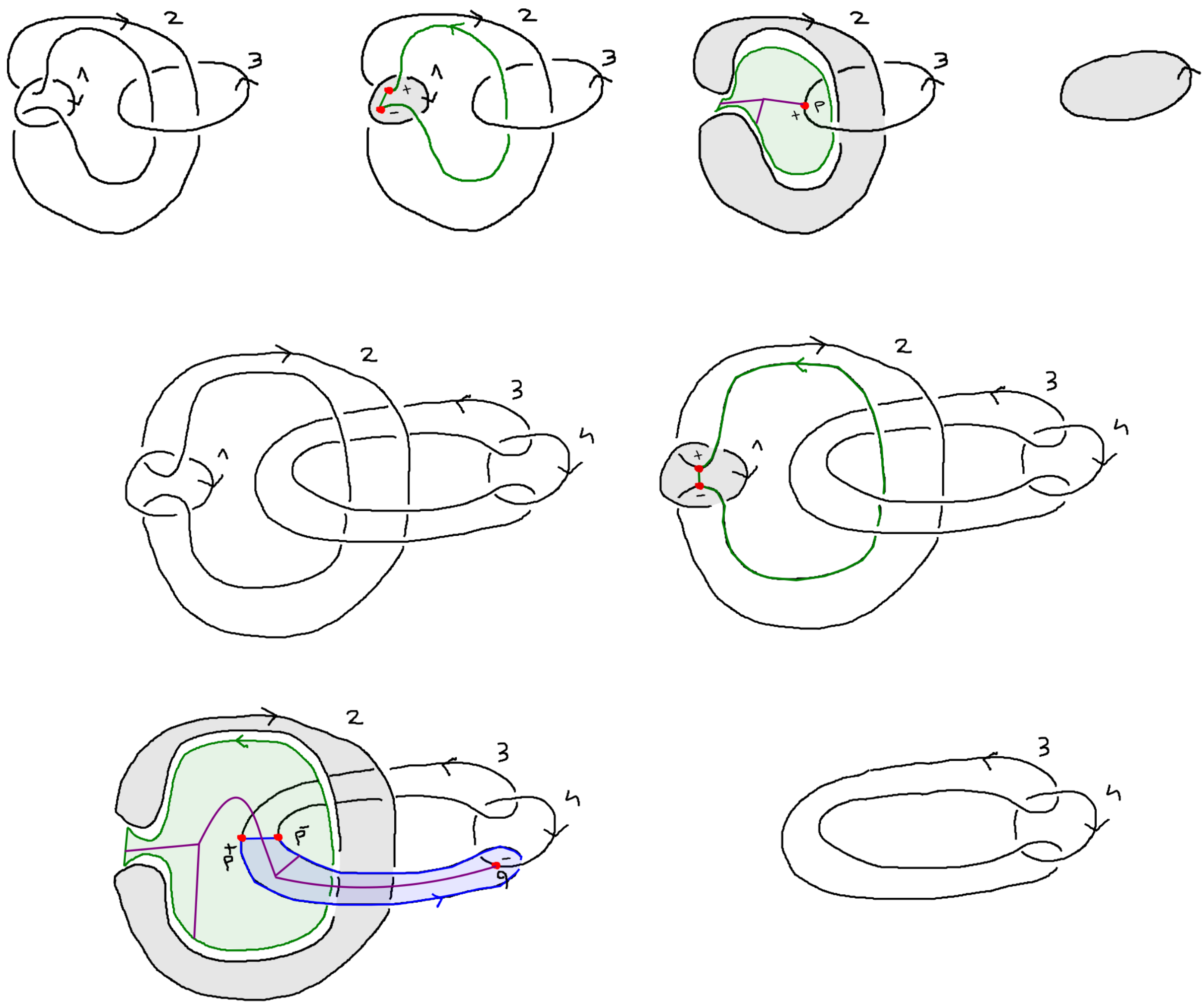

Figure 1.13: Raising obstructions

The tree $c(t')$ is associated to the unique intersection point $p \in W_I \cap L_{l(v_1)}$ of order $n-1$, here W_I (green in Figure 14a) is a Whitney disk of order $n-1$ in $\mathcal{W}'$. By Bing doubling $L_{l(v_1)}$ this component is replaced by two unknots $K_{l(v_1)}, K_{l(v_2)}$ (labels 3 and 4 in Figure 14b) and the required tower is $\mathcal{W} = \mathcal{W}'' \cup \mathcal{W}'''$, here $\mathcal{W}'''$ is $\mathcal{W}'$ with $W_{l(v_1)}$ removed, while $\mathcal{W}''$ is defined as follows. $K_{l(v_1)} \cup K_{l(v_2)}$ can be extended radially inside D^4 while staying in a tubular neighborhood of the old component $L_{l(v_1)}$ in each intermidiate copy of S^3 and this guarantees that the created two annuli $A_{l(v_1)}, A_{l(v_2)}$ intersect $\mathcal{W}'$ exactly in two points p^+, p^- of opposite sign replacing the old unpaired intersection point p. We can arrange $W_I \cap A_{l(v_1)} = \{p^+, p^-\}$ and $\mathcal{W}' \cap A_{l(v_2)} = \emptyset$ and use half of one of the two natural disks coming from the Bing double construction as a Whitney disk W_J (blue in Figure 14b) of order n that pairs $\{p^+, p^-\}$ and generates a new order n unpaired intersection point $q = W_J \cap A_{l(v_2)}$. By further extending $A_{l(v_1)}, A_{l(v_2)}$ radially inside D^4 we finally get an unlink with two components and capping them off with disjoint disks we complete the description of disks of order 0 $W_{l(v_1)}$ and $W_{l(v_2)}$ bounding $K_{l(v_1)}$ and $K_{l(v_2)}$. Then $\mathcal{W}'' = W_{l(v_1)} \cup W_{l(v_2)} \cup W_J$ and one has $\tau_n(\mathcal{W}) = t$. By sum connecting links and towers one can realize arbitrary sums of trees in $\mathcal{T}_n$ with distinct labels.

Finally, if a tree in $\mathcal{T}_n$ has one repeated label l on k vertices, choose colors $c_1, \ldots, c_k$ and relabel those vertices with $(l, c_1), \ldots, (l, c_k)$, now the tree has distinct labels and can be realized by a link L with $m + k - 1$ components and a suitable Whitney tower. Choose $k - 1$ bands connecting the components $L_{(l,c_i)}$ for $1 \leq i \leq k$, performing then band sums of the componets and boundary connected sums of the relative disks of order 0 we get a new link with m components and a new Whitney tower with same intersection tree as before but with identified labels $(l, c_1), \ldots, (l, c_k) = l$. □

1.6 Gropes

We introduce a technical tool of 4-dimensional topology that will be used in the next section to prove the main theorem connecting intersection invariants of Whitney towers and Milnor invariants of links. For more informations on gropes we refer to [**FQ90**].

DEFINITION 3. *A grope G is a 2-complex constructed as follows:*

- *the bottom stage is a compact oriented connected surface of positive genus with one boundary component and comes with a symplectic basis of circles on it*
- *higher stages are obtained by attaching punctured tori with compatible orientations to any number of basis circles in any lower stage that do not already have a torus attached to them, and choosing a symplectic pair of circles on each attached torus*

the basis circles in all stages of G that do not have a torus attached to them are called tips, attaching disks along the tips we get a capped grope G^c. We say that an oriented link $L \subset S^3$ bounds a capped grope $G^c \subset D^4$ if the components L_i bound properly embedded and disjoint gropes $G_i \subset D^4$ with compatible orientations and such that their tips bound caps that are framed disks whose interiors are disjoint, but each one intersects in exactly one point the bottom stage of some grope G_j (pheraps with $i \neq j$).

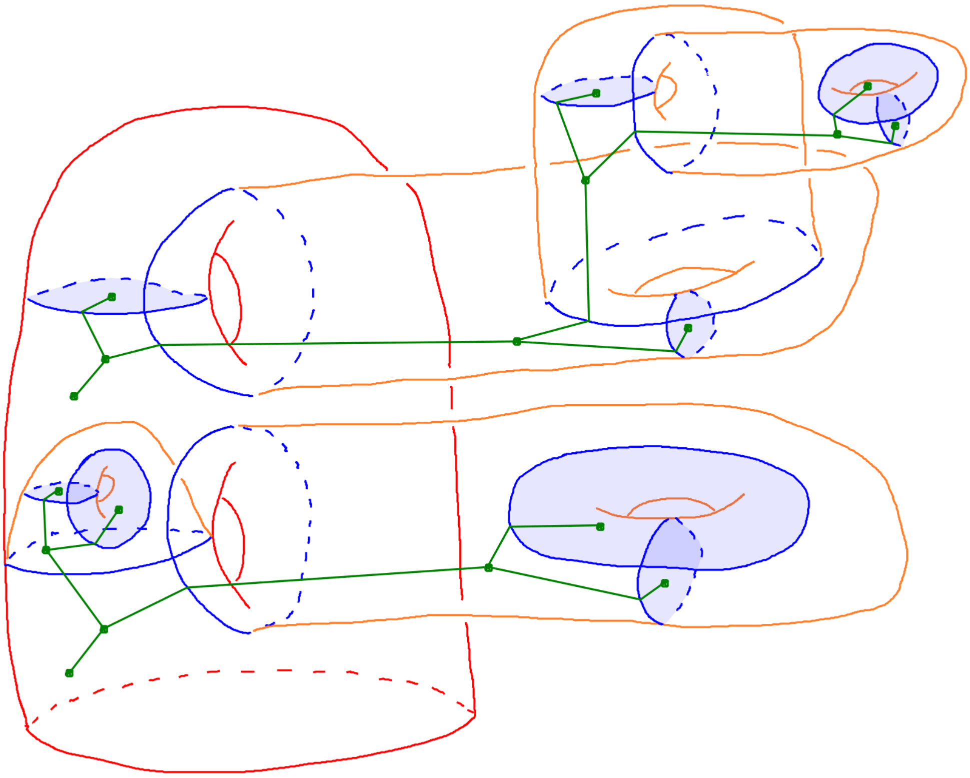

Figure 1.14: Grope

To a capped grope G^c bounding $L \subset S^3$ we associate a combinatorial forest as follows. For each G_i choose g_i embedded arcs in the interior of its bottom stage, where g_i is its genus and we label one of the two vertices of the arc with i. Choose an interior point for each torus in the higher stages of G_i and one for each disk capping a tip, then for every symplectic curve choose a path intersecting it once and connecting two interior points in the regions of G_i glued by that curve (in the case of the bottom stage, one is the unlabelled vertex of an edge).

This procedure gives g_i trees for every G_i, each one with vertices of degree three or one. Vertices of degree one have a natural labelling because by hypothesis each capping disk intersects in exactly one point the bottom stage of some grope G_j. Vertices of degree three have a natural cyclic orientation induced by the grope. Every tree comes with a sign computed as product of signs of intersection points between capping disks and bottom stages of gropes in D^4. Therefore we have a well-defined combinatorial forest $t(G^c)$ obtained as disjoint union of trees coming from G_i for all i. Finally, we call class of G^c the minimum degree of a tree in $t(G^c)$ plus 1.

We sketch below a procedure to convert Whitney towers to gropes, for a more detailed description see [**CST14**].

PROPOSITION 4. *If $L \subset S^3$ oriented link bounds an order $n \geq 1$ Whitney tower $\mathcal{W}$, then it bounds a class $n+1$ capped grope G^c such that $t(\mathcal{W}) \cong t(G^c)$ as graphs (forgetting roots and labels). Furthermore the previous isomorphism can be chosen to preserve signs of tree components.*

Proof. We can use finger moves to make $\mathcal{W}$ split without modifying $t(\mathcal{W})$. Being $n \geq 1$ all order 0 intersection points are paired by Whitney disks and called $p_1, \dots, p_N$ the unpaired intersections points in the tower we have $t(\mathcal{W}) = t_{p_1} \sqcup \dots \sqcup t_{p_N}$ where t_{p_i} is a tree of degree $d_i \geq n$. Each t_{p_i} is embedded in a subtower $\mathcal{W}_{p_i}$ and the idea is to turn this into a branch $G^c_{j,i}$ of a capped grope $G^c_j = \cup_i G^c_{j,i}$ on L_j. The notation for this grope suggests that it comes from a split subtower $\mathcal{W}_{p_i}$ with one order 1 disk pairing two intersection points in the order 0 disk of $\mathcal{W}$ that bounds L_j, see Figure 1.15 for an example with $j = 1$ and p_i of order 3.

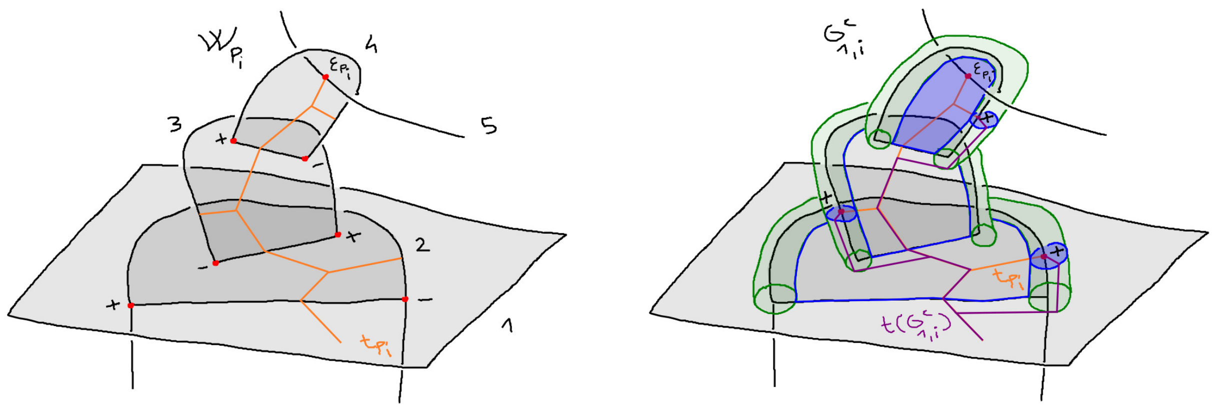

Figure 1.15: Groping simple

Assume first that t_{p_i} is a simple tree, this means it has maximal diameter among trees with same number of vertices. Order the disks in $\mathcal{W}_{p_i}$ by height and call them $W^1, \dots, W^{d_i}$. The normal bundle of W^1 restricts to a disk bundle over ∂W^1, we can use this as a guide to remove two disks D^+, D^- around the intersection points of order zero paired by W^1 and glue in a tube. Performing this surgery on every pair of cancelling intersections the disk W_j of $\mathcal{W}$ becomes a compact connected oriented surface W'_j properly embedded in D^4 and with boundary L_j, this comes with a canonical symplectic basis of circles by taking, for every tube, its intersection with the disk W_1 relative to its tower and a meridian.

These circles also have natural capping disks from W^1 and the normal disks to ∂W^1 respectively. The normal disk intersects a Whitney arc in one point by definition and this point belongs to a disk of order 0 in $\mathcal{W}$ because by assumption t_{p_i} is simple and this says that the disk will be a cap in the final grope $G^c_{j,i}$. The disk coming from W^1 instead could have one intersection point or two, thanks to the fact that $\mathcal{W}$ is split. If it has one point then reasoning as before it's a cap in the final grope $G^c_{j,i}$, $d_i = 1$ and the point must be p_i, so we are done. If it has two points paired by W_2 then we can use this disk to get a higher stage of the grope as before, gluing the tube to the disk gives a punctured torus whose boundary is one of the chosen symplectic circles. Proceeding in this way we finally get to p_i and complete the construction of $G^c_{j,i}$ and $G_j = \cup_i G^c_{j,i}$ are then the higher stages of a grope on L_j whose bottom stage is W'_j.

These gropes are disjointly embedded because boundaries of Whitney disks are disjointly embedded by definition of Whitney tower. From this definition also follows that all Whitney disks are framed, thus Whitney sections (see Chapter 2) give a set of circles homologous to chosen symplectic circles in the gropes and we can use these as a guide to build a forest

$t(G^c) \cong t(\mathcal{W})$ as required, this is clear from Figure 1.15.

Furthermore we can choose orientations on the caps of the gropes in such a way that they intersect positively the bottom stages, except for caps intersecting them in original unpaired intersection points p_i that still intersect with the original sign. This guarantees that the sign of each tree in $t(G^c)$ equals the sign of the corresponding tree in $t(\mathcal{W})$.

To conclude the proof, we have to deal with the case of a non-simple tree t_{p_i}. The previous surgery process goes upward along a tree t_{p_i} until we have to deal with a Whitney disk W in $\mathcal{W}_{p_i}$ that has a single intersection point q with another disk W_K in $\mathcal{W}$. In the previous case W_K had to be a disk of order 0 and gave rise to a cap in the final grope, but if W_K has positive order we proceed as follows.

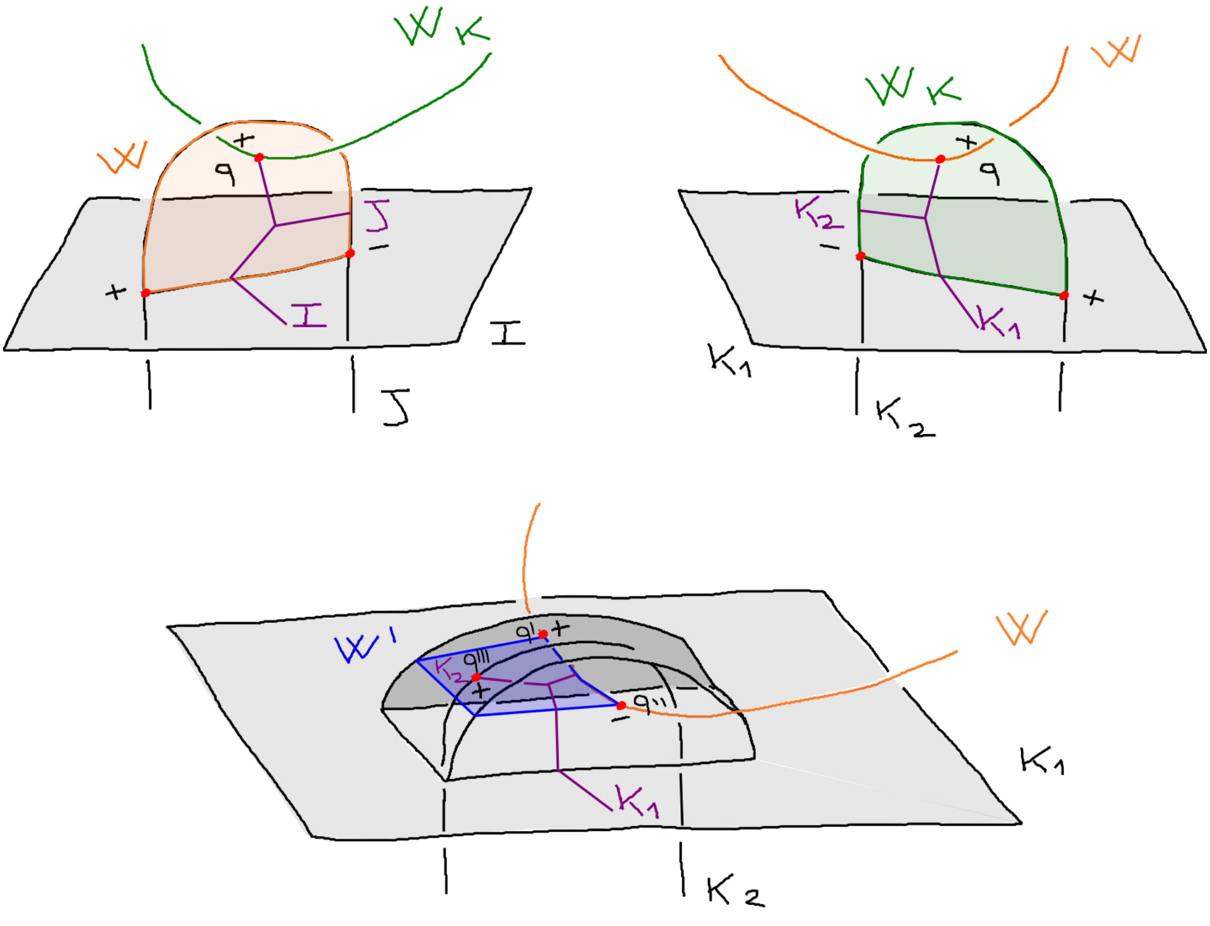

Figure 1.16: Groping general

Let $K = (K_1, K_2)$ be the tree associated to the Whitney disk W_K as in Figure 1.16b. We can use W_K to perform a Whitney move that removes the intersections paired by W_K and $q = W \cap W_K$ but creates two new intersections $\{q', q''\} = W \cap K_1$ paired by a new Whitney disk W' and $W' \cap K_2 = q'''$. This gives a new Whitney tower $\mathcal{W}'$ with W' instead of W_K and Figure 1.16c shows that $t(\mathcal{W}') \cong t(\mathcal{W})$ as signed graphs. One can repeat this procedure to ensure that a single intersection point $q \in W$ always comes from an intersection of W with a disk of order 0 in $\mathcal{W}$ and therefore it gives rise to a cap in the final grope G^c.

Finally, G^c is of class $n+1$ because each tree in $t(G^c)$ has order $d_i \geq n$ and there is at least one of order n being $\mathcal{W}$ a Whitney tower of order n. $\square$

1.7 Connection with Milnor invariants

Once we have a satisfying obstruction theory for Whitney towers, we can explore the connection between intersection invariants and other invariants depending only on the link data. We start with Milnor invariants, see [**Mil54, Mil57**] for the original definition of these invariants.

Assume that all the longitudes of $L \subset S^3$ lie in the $(n+1)$-term of the lower central series of the link group $\pi_1(S^3 \setminus L)_{n+1}$, then a choice of meridians $\alpha_1, \dots, \alpha_m$ induces an isomorphism

$$\frac{\pi_1(S^3 \setminus L)_{n+1}}{\pi_1(S^3 \setminus L)_{n+2}} \cong \frac{\mathsf{F}_{n+1}}{\mathsf{F}_{n+2}}$$

where F is the free group on m generators α_i. The group on the right is clearly abelian and, by interpreting commutators as Lie brackets, it is isomorphic as abelian group to the degree $n+1$ part of the free Lie algebra on m generators $\mathsf{L}(m) = \bigoplus_n \mathsf{L}_n$ (grading given by commutator lenghts).

DEFINITION 4. *If $L \subset S^3$ oriented link, choose a base point in $S^3 \setminus L$, meridians α_i and longitudes l_i, then if $l_i \in \pi_1(S^3 \setminus L)_{n+1}$ for all i the order n universal Milnor invariant of L is defined as*

$$\mu_n(L) = \sum \alpha_i \otimes l_i \in \mathsf{L}_1 \otimes \mathsf{L}_{n+1}$$

Milnor's work shows that $\mu_n(L)$ only depends on the isotopy class of L and is thus a well defined link invariant (as long as the condition on longitudes is satisfied).

We explain below in the framed setting one of the main results of [**CST14**], which also holds in the twisted case as discussed in Chapter 2.

THEOREM 3. *If $L \subset S^3$ oriented link bounds a Whitney tower $\mathcal{W}$ of order $n \geq 1$, then $\mu_k(L) = 0$ for $k < n$ and*

$$\mu_n(L) = \eta_n(\tau_n(\mathcal{W}))$$

here $\eta_n : \mathcal{T}_n \to \mathsf{L}_1 \otimes \mathsf{L}_{n+1}$ is a group morphism that turns trees into brackets given by the formula

$$\eta_n(t) = \sum_v \alpha_{l(v)} \otimes B_v(t)$$

where v ranges among degree 1 vertices of t, we call $l(v)$ their labels, α_i freely generate L and $B_v(t) \in \mathsf{L}_{n+1}$ is the obvious bracketing associated to the tree t once we choose v as root.

Proof. We give a proof for $n = 1$, the ideas involved in the $n > 1$ case are similar. Intersection points of order 0 in $\mathcal{W}$ are paired by disks of order 1 and therefore all linking numbers and framings of L are 0 and this implies $\mu_0(L) = 0$. Then chosen α_i meridians and l_i longitudes for L we have $l_i \in \pi_1(S^3 \setminus L)_2$ for every i and

$$\mu_1(L) = \alpha_1 \otimes l_1 + \ldots + \alpha_m \otimes l_m \in \mathsf{L}_1 \otimes \mathsf{L}_2$$

On the other hand $\tau_1(\mathcal{W}) = \epsilon_{p_1} t_{p_1} + \ldots + \epsilon_{p_k} t_{p_k}$ with p_s unpaired intersection of order 1 and t_{p_s} associated tree of degree 1 and called t_{p_s} the tripod with labels $a_s, b_s, c_s \in \{1, \ldots, m\}$ we have

$$\eta_1(\tau_1(\mathcal{W})) = \eta_1(\sum_{s=1}^{k} \epsilon_{p_s} t_{p_s}) = \sum_{s=1}^{k} (\alpha_{a_s} \otimes [\alpha_{c_s}, \alpha_{b_s}]^{\epsilon_{ps}} + \alpha_{b_s} \otimes [\alpha_{a_s}, \alpha_{c_s}]^{\epsilon_{ps}} + \alpha_{c_s} \otimes [\alpha_{b_s}, \alpha_{a_s}]^{\epsilon_{ps}}) =$$

$$= \sum_{i=1}^{m} (\alpha_i \otimes \prod_{1 \le s \le k, a_s = i} [\alpha_{c_s}, \alpha_{b_s}]^{\epsilon_{ps}} + \alpha_i \otimes \prod_{1 \le s \le k, b_s = i} [\alpha_{a_s}, \alpha_{c_s}]^{\epsilon_{ps}} + \alpha_i \otimes \prod_{1 \le s \le k, c_s = i} [\alpha_{b_s}, \alpha_{a_s}]^{\epsilon_{ps}})$$

The last equality holds because (a summand begins with $\alpha_i \otimes$) $\iff$ ($a_s = i$ or $b_s = i$ or $c_s = i$) and if some tree t_{p_s} has a repeated label i then the corresponding three terms in the last expression sum to 0. We can rewrite the previous in compact form as

$$\sum_{i=1}^{m} \alpha_i \otimes (\prod_{1 \le s \le k} [\alpha_{c_s}, \alpha_{b_s}]^{\epsilon_{ps} \delta_{ia_s}} [\alpha_{a_s}, \alpha_{c_s}]^{\epsilon_{ps} \delta_{ib_s}} [\alpha_{b_s}, \alpha_{a_s}]^{\epsilon_{ps} \delta_{ic_s}})$$

Now it suffices to show that for every i one has

$$l_i = \prod_{1 \le s \le k} [\alpha_{c_s}, \alpha_{b_s}]^{\epsilon_{ps} \delta_{ia_s}} [\alpha_{a_s}, \alpha_{c_s}]^{\epsilon_{ps} \delta_{ib_s}} [\alpha_{b_s}, \alpha_{a_s}]^{\epsilon_{ps} \delta_{ic_s}} \in \mathsf{L}_2 \cong \frac{\pi_1(S^3 \setminus L)_2}{\pi_1(S^3 \setminus L)_3}$$

Figure 1.17 shows how the boundary of a punctured genus g surfacace is a product of commutators of symplectic pairs of circles. Then it's natural to try to show that each l_i bounds a punctured surface realizing the previous identity.

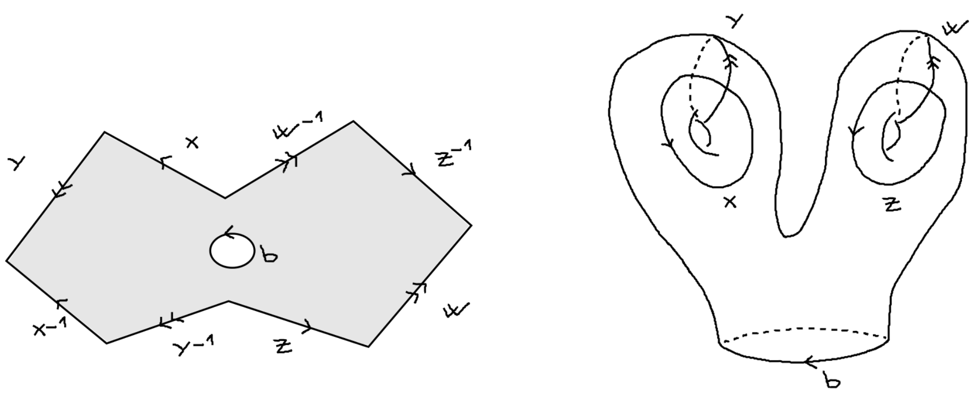

Figure 1.17: Commutators

We use finger moves to make $\mathcal{W}$ split and take a capped grope G^c of class 2 bounding L and with same intersection invariant as described in Proposition 4. This grope has a surface G_i of genus g_i bounding L_i for every i and these are disjointly embedded in D^4. Furthermore every surface has a symplectic system of $2g_i$ circles capped by disks intersecting some G_j exactly once and for each pair of disks in G^c there is an embedded copy of $t_{p_s} \subset G^c$ with labels a_s, b_s, c_s. Now we modify each G_i to get a puncutred surface Σ_i bounding L_i and with genus equal to r number of vertices with label i in $\sqcup t_{p_s}$, see Figure 1.18.

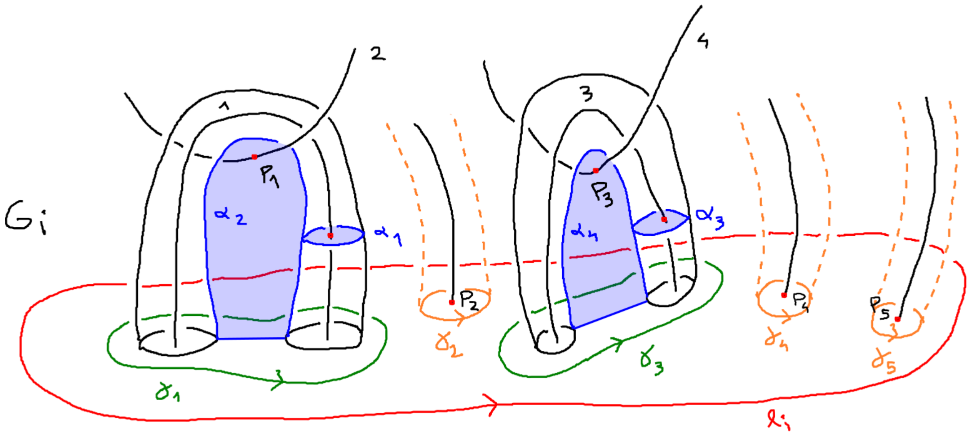

Figure 1.18: Surgered bottom stage

A parallel push-off of G_i bounds $l_i \subset S^3$ and $l_i = \gamma_{i_1} \cdots \gamma_{i_r}$ in $\pi_1(D^4 \setminus G^c)$, i.e. it's a product of embedded loops in a parallel push-off of G^c with each loop bounding either a branch of the grope (type 1, green in Figure 1.18) or a normal disk to a cap intersecting the bottom stage of G_i (type 2, orange in Figure 1.18). Each loop of type 1 bounds a punctured torus in the grope and its contribution to l_i is a commutator of the corresponding symplectic circles. The surface Σ_i is then obtained by removing the disks bounded by loops of type 2 and gluing punctured tori embedded in $D^4 \setminus G^c$ that are dual to caps as explained below.

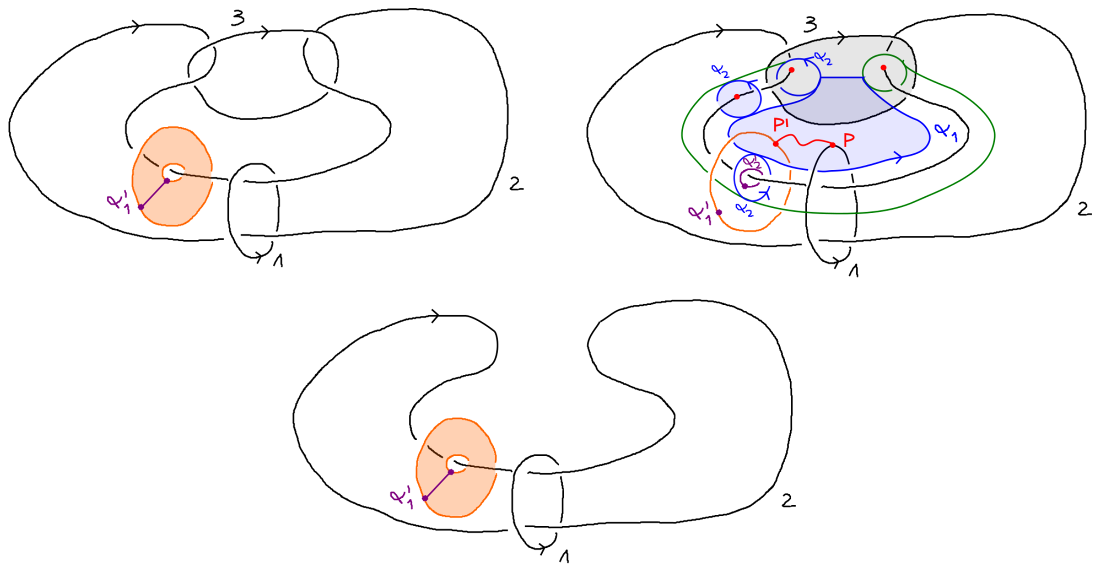

Figure 1.19: Dual torus

We describe a dual torus to a cap c as a circle bundle over a circle, see Figure 1.19 for an example. The base circle of the bundle is any meridian parallel to the other circle in the symplectic pair containing ∂c and intersecting G^c in exactly one point of ∂c (α_2 in the example). The fibers are boundaries of disks in the normal bundle of the stage of G capped by c (green tube in the example) restricted to the base circle chosen before. This torus T is embedded in D^4 and intersects G^c in exactly one point $p' \in c$. Call now $p \in c$ the unique point of intersection of c with a surface G_j in the bottom stage of G given by definition of grope (for example the points p_2, p_4, p_5 in Figure 1.18) and choose an embedded path $u \subset c$ connecting p and p'. The normal bundle of $c \subset D^4$ restricted to u guides a connected sum of T with the disk in G_j bounded by the loop γ of type 2 around $p = c \cap G_j$ (for example the loops $\gamma_2, \gamma_4, \gamma_5$ in Figure 1.18). This connected sum T' is a punctured torus bounding a type 2 loop γ and such that $T' \cap G^c = \gamma$. By construction, this torus comes with two canonical symplectic circles given by the base circle of the bundle and a fiber, and from Figure 1.19 is clear that they are homotopic in $D^4 \setminus G^c$ respectively to the dual circle of ∂c and a meridian of to the boundary of the stage of G capped by c, so that γ is a commutator of the two in $\pi_1(D^4 \setminus G^c)$ with an appropriate sign given by ϵ_p.

The proof ends with the observation that the inclusion $S^3 \setminus L \hookrightarrow D^4 \setminus G^c$ induces

$$\frac{\pi_1(D^4 \setminus G^c)_{n+1}}{\pi_1(D^4 \setminus G^c)_{n+2}} \cong \frac{\pi_1(S^3 \setminus L)_{n+1}}{\pi_1(S^3 \setminus L)_{n+2}}$$

so the expansion in terms of commutators for l_i actually holds in the group on the right as required. See [**Dwy75**] for more details about this isomorphism. □

CHAPTER 2

Twisting phenomena

2.1 Framing obstruction

We discuss here in a general setting the obstruction for a Whitney disk to be framed in dimension 4, this is one of the basic requirements for a successful Whitney move, see [**Sco05**] for more details.

Let X be a smooth oriented 4-manifold and $\Sigma_1, \Sigma_2 \subset X$ two oriented surfaces in generic position intersecting in two points x', x'' with opposite signs (or two sheets of the same immersed surface self-intersecting in these two points). Call $\phi : D^2 \to X$ an immersed Whitney disk W pairing the intersections, then $S^1 = \alpha \cup \beta$ arcs with $\alpha \cap \beta = \{p', p''\}$, $\phi(p') = x'$ and $\phi(p'') = x''$, $\phi(\alpha) \subset \Sigma_1$ and $\phi(\beta) \subset \Sigma_2$ embeddings. The normal bundle of ϕ is a 2-plane bundle $\nu(\phi) \to D^2$ whose fiber over $q \in D^2$ is given by $T_{\phi(q)}X/\operatorname{im}(\mathrm{d}_q\,\phi)$ and being a bundle over a contractible base is trivial and hence its restriction $\nu(\phi)_{|S^1}$ is trivial too.

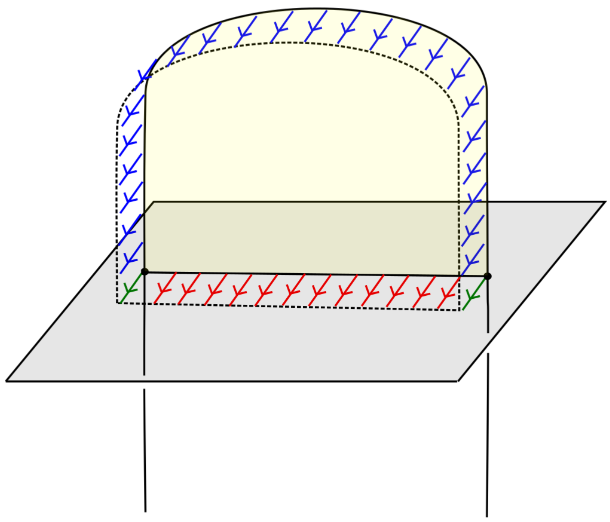

Figure 2.1: Whitney section

Choose two sections w_α, w_β of $\nu(\phi)_{|S^1}$ along α, β with the following properties:

- w_α and w_β have no zeros

- w_α is tangent to Σ_1
- w_β is transverse to Σ_2
- w_α and w_β match in p', p''

The first condition guarantees that the two sections are transversal to the Whitney disk and together with the second and third condition can always be fulfilled as shown in Figure 2.1. The fourth condition is more subtle and can only be satisfied if x' and x'' have opposite signs.

These two sections glue to give a global section w of $\nu(\phi)_{|S^1}$ called Whitney section because of its special properties. This section extends to a nowhere vanishing section over D^2 if and only if $[w] = 0$ in $\pi_1(\mathrm{Gr}_1(\mathbb{R}^2)) \cong \mathbb{Z}$. We point out that the Whitney section has always an extension to D^2 because $\pi_1(\mathbb{R}^2) = 0$ but this could have zeros in the interior and this leads to a failure of the Whitney move.

We call $[w] = \omega(W) \in \pi_1(\mathrm{Gr}_1(\mathbb{R}^2))$ framing obstruction of the Whitney disk W and when $\omega(W) \neq 0$ we say that the disk is twisted. We identify these classes with integer numbers via the group isomorphism

$$\pi_1(\mathrm{Gr}_1(\mathbb{R}^2)) \to \mathbb{Z}$$
$$[s] \mapsto \#(\mathrm{int}(W) \cap \tilde{s}(\mathrm{int}(W))$$

where $\tilde{s}$ is a generic extension of s to the interior of W (some authors call this integer relative Euler number of the normal bundle of W). An argument along the lines of Proposition 2 in Chapter 1 shows that if ∂W and s are in the same spherical slice of D^4 then $\#(\mathrm{int}(W) \cap \tilde{s}(\mathrm{int}(W)) = \mathrm{lk}(\partial W, s)$, where the linking number is measured in that spherical slice.

Examples of twisted Whitney disks with arbitary framing obstruction can be produced via twisted Bing doubling of a component K in a link. This operation consists in performing an ordinary Bing doubling as explained in Section 1.3 but using an embedded circle K' in the boundary $\partial N(K)$ of a tubular neighborhood of the component K with $\mathrm{lk}(K, K') \neq 0$ instead of the usual longitude $K' = l$ with $\mathrm{lk}(K, l) = 0$.

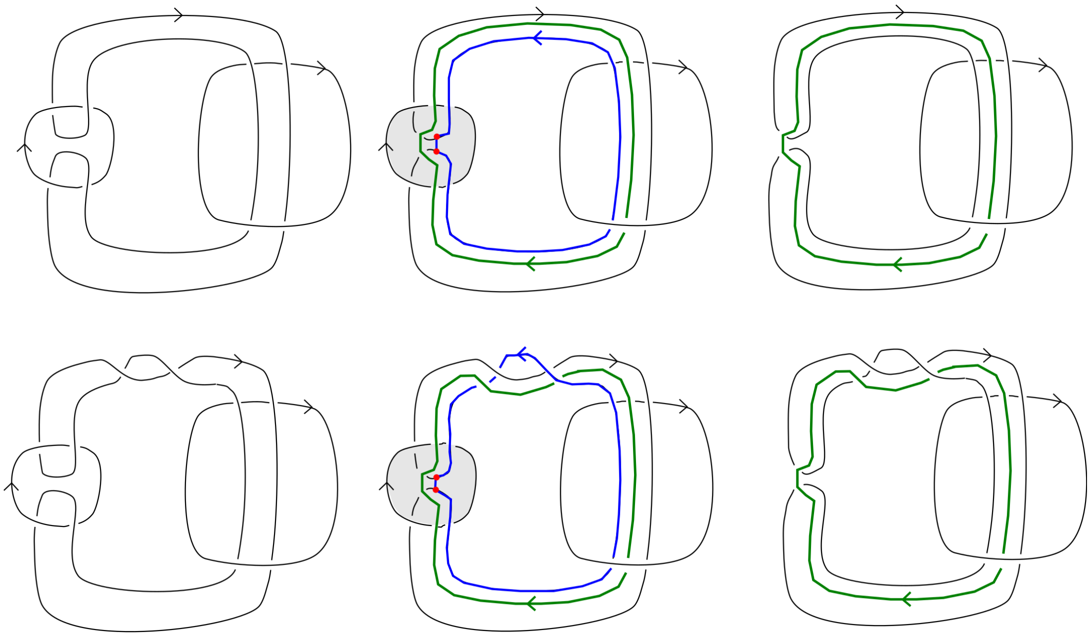

Figure 2.2: Framed and twisted Bing doubles of Hopf link

Figure 2.2 compares two Bing doubles of the Hopf link, the first is the usual untwisted Bing double and the second is a twisted version. In both pictures the blue curve is the boundary of a Whitney disk and the green curve is the boundary of a collar. The framing obstruction is computed by their linking number in the spherical slice of D^4 where they both live.

We describe now a way to kill the framing obstruction at the cost of introducing a new intersection point on the Whitney disk, this operation is called boundary twist or spinning of the Whitney disk.

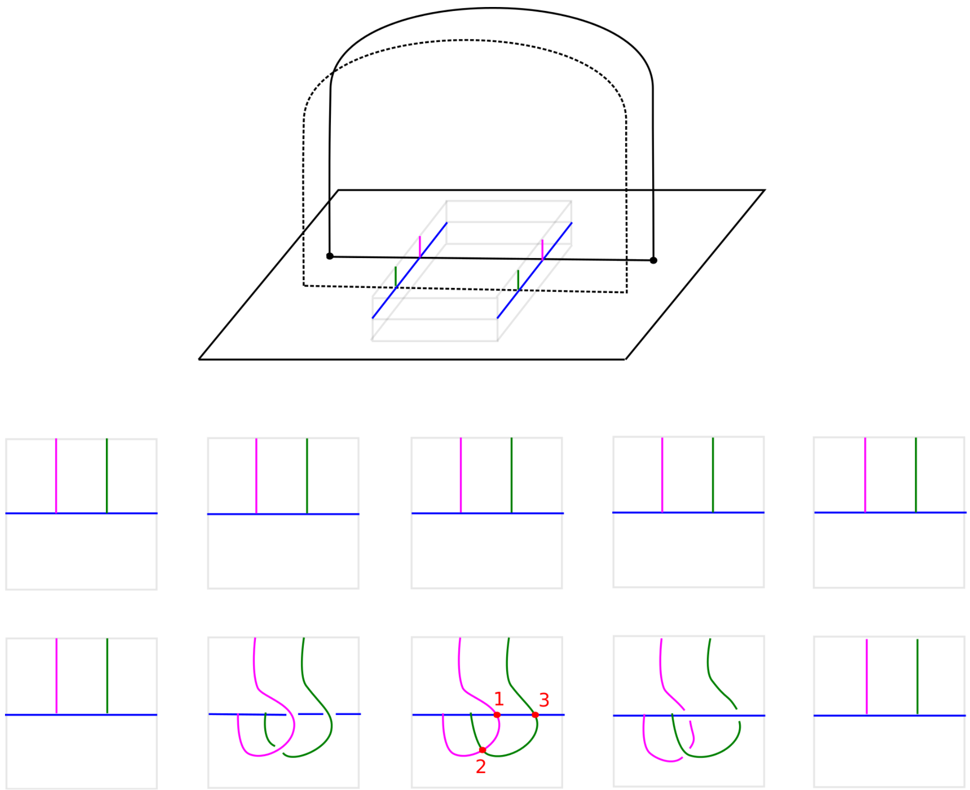

Figure 2.3: Boundary twist

Consider a Whitney disk with $\partial W = \alpha \cup \beta$ paring two sheets Σ_1, Σ_2 in the ambient 4-manifold X and choose a small interior arc $\alpha' \subset \alpha$. The normal bundle $N(\alpha') \subset X$ is a trivial bundle diffeomorphic to $D^3 \times [0,1]$ and the movie in Figure 2.3 shows its intersection with $W \cup w(W) \cup \Sigma_1$, where $w(W)$ is the image of W under any extension of its Whitney section to the whole disk. Notice that such extension could have zeros in the interior but we can shrink $N(\alpha')$ to have $N(\alpha') \cap W \cap w(W) = \emptyset$ and thus we don't see intersection points between W and $w(W)$ in the movie.

The boundary twist consists in removing $\text{int}(N(\alpha'))$ and gluing in a new 4-ball $D^3 \times I$ given by the movie in Figure 2.3. In the middle picture of the movie we see that the new violet Whitney disk W' has exactly one new intersection point with the blue sheet Σ_1 (red dot number 1) and the green image of the Whitney section $w(W')$ also gains such a new intersection with the same sign (red dot number 3). The red dot number 2 is a newly created intersecion point in $W' \cap w(W')$ and it modifies the framing obstruction $\omega(W') = \omega(W) \pm 1$ according to its sign.

If $\omega(W) = k$ we can therefore arrange it to have $k = 0$ up to introducing $|k|$ new intersection points in $W \cap \Sigma_1$. We point out that the same costruction can be carried on the arc $\beta \subset W \cap \Sigma_2$ as well and thus we are free to distribute the new intersection points of W among Σ_1 and Σ_2.

2.2 4d interpretation of Arf invariant

The framing obstruction is related to the Arf invariant of knots. If $K \subset S^3$ knot and F Seifert surface, $\mathrm{Arf}(K) \in \mathbb{Z}/2\mathbb{Z}$ is defined as the Arf invariant of the quadratic form over $\mathbb{Z}/2\mathbb{Z}$ associated to the bilinear Seifert form on $H_1(F;\mathbb{Z})$ reduced modulo 2. Figure 2.4 shows how to expand this invariant as

$$\mathrm{Arf}(K) \equiv \sum_{i=1}^{g(F)} \mathrm{lk}(\alpha_i^*, \alpha_i)\,\mathrm{lk}(\beta_i^*, \beta_i) \pmod 2$$

where $\{\alpha_i, \beta_i\}$ is a symplectic basis of curves for $H_1(F;\mathbb{Z})$ and * denotes their images under an outward nowhere vanishing field normal to F.

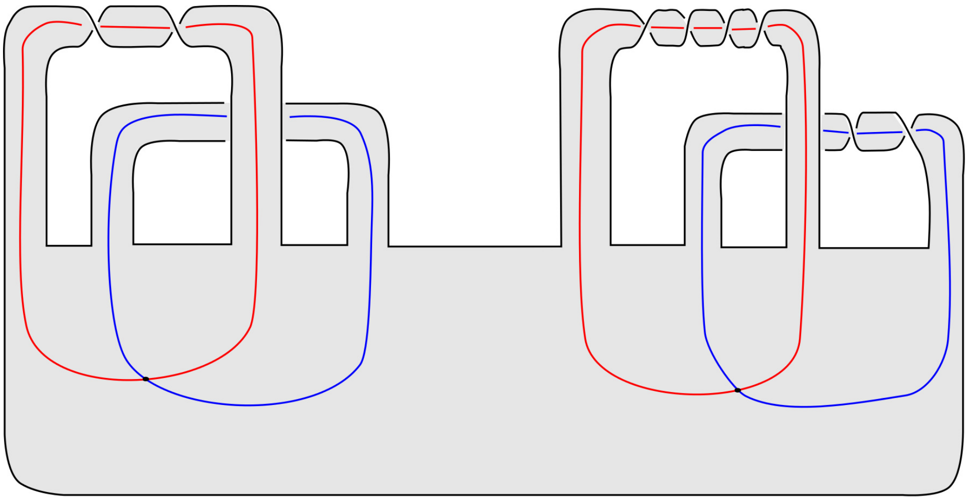

Figure 2.4: Arf invariant

This is a well defined invariant of knots because it does not depend on the chosen Seifert surface F, in fact two different Seifert surfaces have S-equivalent Seifert forms, see [**Kau87**] for more about this.

PROPOSITION 5. *Any knot $K \subset S^3$ bounds a Whitney tower of order 2 by allowing twisted order 1 disks, furthermore for any such tower*

$$\mathrm{Arf}(K) \equiv \sum_i \omega(W_i) \pmod 2$$

where the sum ranges over W_i order 1 disks in the tower.

Proof. Thanks to Proposition 1 of Chapter 1 we know that K bounds a generically immersed disk $D \subset D^4$ and this will be the only order 0 disk of the tower. To get a tower of order 2 we need to pair self-intersections of D with order 1 Whitney disks, a priori this is not possible because D could have a different number of positive and negative self-intersections, but we can fix this by adding kinks as decribed below.

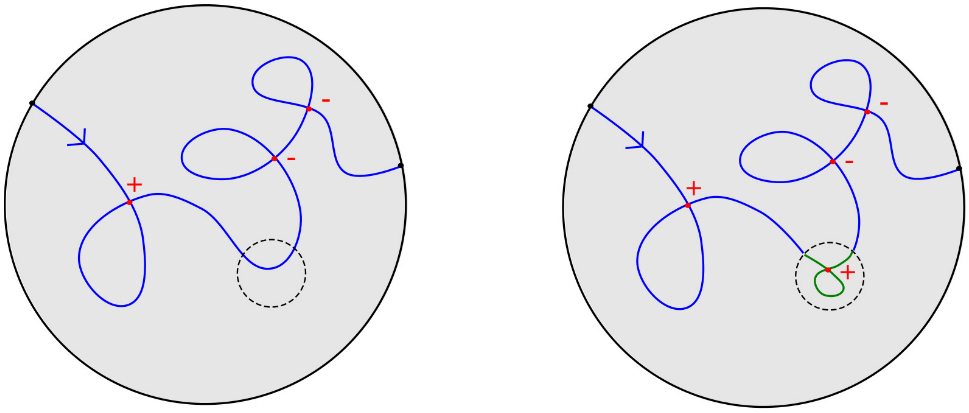

Figure 2.5: Adding kinks

A schematic picture in half the dimension is shown in Figure 2.5. More precisely, one can remove the interior of a small 4-ball $B(p)$ centered in a point of $p \in D$ that is not of self-intersection and glue in a new 4-ball $B'(p)$ described as follows. $B'(p)$ contains in its interior a smaller 4-ball $B''(p)$ with two properly embedded disks intersecting in p with sign ± 1 and whose boundaries in $\partial B''(p)$ form a Hopf link. This link bounds an annulus in $\partial B''(p)$ and removing an open disk in the interior of this annulus we create a new unknotted boundary component that can be glued with another annulus properly embedded in $B'(p) \setminus \mathrm{int}(B''(p))$ to the unknot $\partial B(p) \cap D \subset \partial B(p) = \partial B'(p)$.

We can now choose $W_1, \dots, W_k$ Whitney disks pairing order 0 intersections, these may have nontrivial framing obstruction and thus we get a twisted Whitney tower of order 1. Thanks to Proposition 4 of Chapter 1 this tower can be converted to a capped grope of class 2, whose caps are now not necessarily framed or embedded because they come from order 1 Whitney disks W_i that can be twisted and generically immersed.

The bottom stage of this grope is a properly embedded connected surface of genus k in D^4 whose boundary is $K \subset S^3$, for each $1 \leq i \leq k$ we have a caps C_i', C_i'' bounding circles of the symplectic basis α_i, β_i as in Figure 2.6

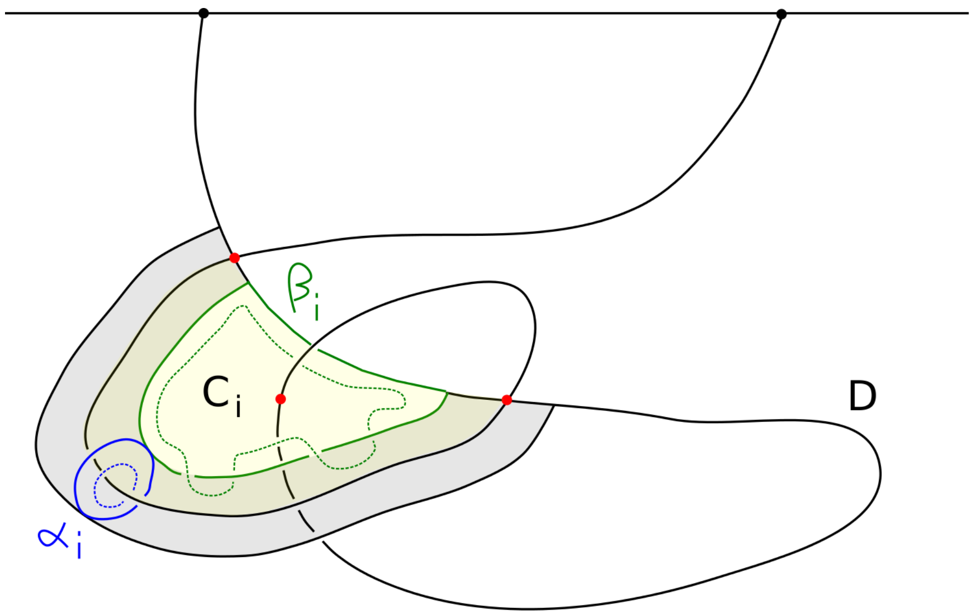

Figure 2.6: Twisted cap

We have

$$\mathrm{Arf}(K) \equiv \sum_{i=1}^{k} \mathrm{lk}(\alpha_i, \alpha_i^*)\,\mathrm{lk}(\beta_i, \beta_i^*) \equiv \sum_{i=1}^{k} (\#(\mathrm{int}(C_i') \cap D) + \omega(C_i'))(\#(\mathrm{int}(C_i'') \cap D) + \omega(C_i''))$$

Now C_i' intersects D in exactly one point and it's framed thanks to how the groping procedure works, C_i'' instead intersects D in $\#(\mathrm{int}(W_i) \cap D)$ and has framing $\omega(C_i'') = \omega(W_i)$, therefore

$$\mathrm{Arf}(K) \equiv \sum_{i=1}^{k} (\#(\mathrm{int}(W_i) \cap D) + \omega(W_i))$$

Now performing boundary twists as described in the previous section we can kill $\#(\mathrm{int}(W_i) \cap D)$ while modifying $\omega(W_i)$ but their sum remains the same modulo 2 and we get

$$\mathrm{Arf}(K) \equiv \sum_{i=1}^{k} \omega(W_i)$$

and $\#(\mathrm{int}(W_i) \cap D) = 0$ tells that we can pair order 1 intersections with order 2 Whitney disks, this gives the required twisted Whitney tower. $\square$

2.3 Twisted Whitney towers

Proposition 5 in the previous section suggests a new definition of Whitney tower in which the framing condition is dropped. We call such towers twisted.

The general philosophy is that the goals of embedding and framing disks contrast each other and we can use boundary twists to either kill framing obstructions while introducing new intersections or to create cancelling mates for unpaired intersection points at the price of loosing framing on disks.

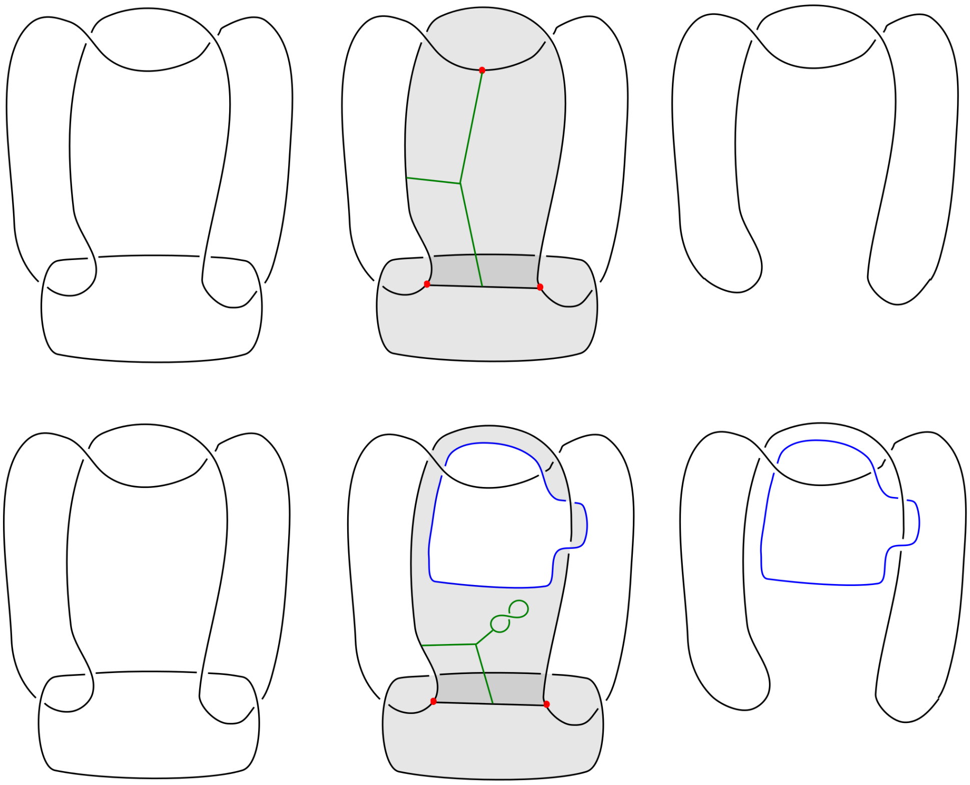

Figure 2.7: Twisted tower on Whitehead link

Figure 2.7 describes a Whitney tower $\mathcal{W}$ of order 1 on the Whitehead link and according to Theorem 1 of Chapter 1 this cannot be raised as $\tau_1(\mathcal{W}) \neq 0$. Nevertheless if we replace the order 1 disk with a twisted Whitney disk we get a new twisted Whitney tower $\mathcal{W}'$ of higher order.

This example motivates the following extension of the obstruction theory of Chapter 1. A twisted Whitney disk pairing sheets whose intersection trees are I and J has an associated twisted tree

$$(I,J)^\infty := \infty \,-\!\!<^{I}_{J}$$

that keeps track of the fact that we cannot perform a successful Whitney move on that twisted disk.

We define below a group of obstructions $\mathcal{T}^\infty$ analogous to $\mathcal{T}$ in the framed setting of Chapter 1. For a discussion on the meaning of its relations see [**CST12**].

DEFINITION 5. *If $n = 2k+1$ then $\mathcal{T}^\infty_{2k+1}$ is the quotient of $\mathcal{T}_{2k+1}$ by the subgroup generated by*

$$i \,-\!\!<^{J}_{J}$$

where J ranges over rooted trees of order k.
If $n = 2k$ then $\mathcal{T}^\infty_{2k}$ is the free abelian group on trees of order $2k$ and ∞-trees of the form

$$J^\infty := \infty\!-\!\!-J$$

where J ranges over rooted trees of order k, modulo the following relations:

- *AS and IHX relations on order $2k$ trees non containing the label ∞*
- *twisted symmetry relation $(-J)^\infty = J^\infty$*
- *twisted IHX relation $I^\infty = H^\infty + X^\infty - \langle H, X\rangle$*
- *interior twist relation $2J^\infty = \langle J, J\rangle$*

Given $\mathcal{W}$ twisted Whitney tower of order n its intersection invariant is defined as

$$\tau_n^\infty(\mathcal{W}) = \sum \epsilon_p t_p + \sum \omega(W_J) J^\infty \in \mathcal{T}_n^\infty$$

where the first sum is over all unpaired intersection points of order n and the second sum is over all order $n/2$ Whitney disks W_J with twisting $\omega(W_J) \in \mathbb{Z}$.

The following theorem, whose proof can be found in [**CST12**], says that these are in fact complete obstructions and is the analogous of Theorem 1 of Chapter 1 in the twisted setting.

THEOREM 4. *Given $L \subset S^3$ oriented link and $\mathcal{W}$ order n twisted Whitney tower on it then $\tau_n^\infty(\mathcal{W}) = 0$ if and only if L bounds an order $n+1$ twisted Whitney tower.*

2.4 Concordance filtration

DEFINITION 6. *For $n \geq 1$, two oriented framed links L_0, L_1 are Whitney concordant of order n if the i-th components of $L_0 \subset S^3 \times \{0\}$ and $-L_1 \subset S^3 \times \{1\}$ cobound a properly immersed annulus $A_i \subset S^3 \times [0,1]$ for each i with the annuli in generic position and $\cup A_i$ supporting a twisted Whitney tower of order n.*

Fixed m number of components, the set $\mathbb{L}$ of m-links has a filtration

$$\cdots \subseteq \mathbb{W}_3^\infty \subseteq \mathbb{W}_2^\infty \subseteq \mathbb{W}_1^\infty \subseteq \mathbb{L}$$

where $\mathbb{W}_n^\infty$ is the set of those bounding a twisted Whitney tower of order n. We denote with W_n^∞ the quotient of $\mathbb{W}_n^\infty$ modulo the equivalence relation given by Whitney concordance of order $n+1$.

PROPOSITION 6. *Band sum # defines a group operation on W_n^∞ and this is abelian and finitely generated.*

Proof. The operation of band sum on oriented m-links was used in Theorem 2 of Chapter 1, we give here a more precise description. If $L_1, L_2 \subset S^3$ oriented links with m components, form S^3 as connected sum of two 3-spheres containing L_1 and L_2 along balls in their complements. Let β be a collection of m disjoint embedded bands joining the components of L_1 to those of L_2 bijectively and oriented compatibly with the link orientations. Then $L_1 \#_\beta L_2 \subset S^3$ is the band-wise connected sum of each pair of components.

In contrast with the case of knots, for $m > 1$ the concordance class of $L_1 \#_\beta L_2$ depends on β but it turns out that it doesn't on W_n^∞ and we can then drop β. In fact, if $L_1', L_2' \subset S^3$ are two other oriented m-links with $[L_i] = [L_i'] \in \mathsf{W}_n^\infty$ then by definition of W_n^∞ they cobound a family of properly immersed annuli in $S^3 \times [0,1]$ supporting an order $n+1$

twisted Whitney tower $\mathcal{V}$ and taken $\mathcal{W}_i$ order n twisted Whitney tower bounding L_i in D^4 this can be extended by $\mathcal{V}$ to form an order n twisted Whtiney tower $\mathcal{W}'_i$ in D^4 on L'_i and $\tau_n^\infty(\mathcal{W}'_i) = \tau_n^\infty(\mathcal{W}_i) + \tau_n^\infty(\mathcal{V}) = \tau_n^\infty(\mathcal{W}_i)$ because $\tau_n^\infty(\mathcal{V}) = 0$ by Theorem 4 in the previous section. Now we want to show that $[L_1 \#_\beta L_2] = [L'_1 \#_{\beta'} L'_2] \in \mathsf{W}_n^\infty$.

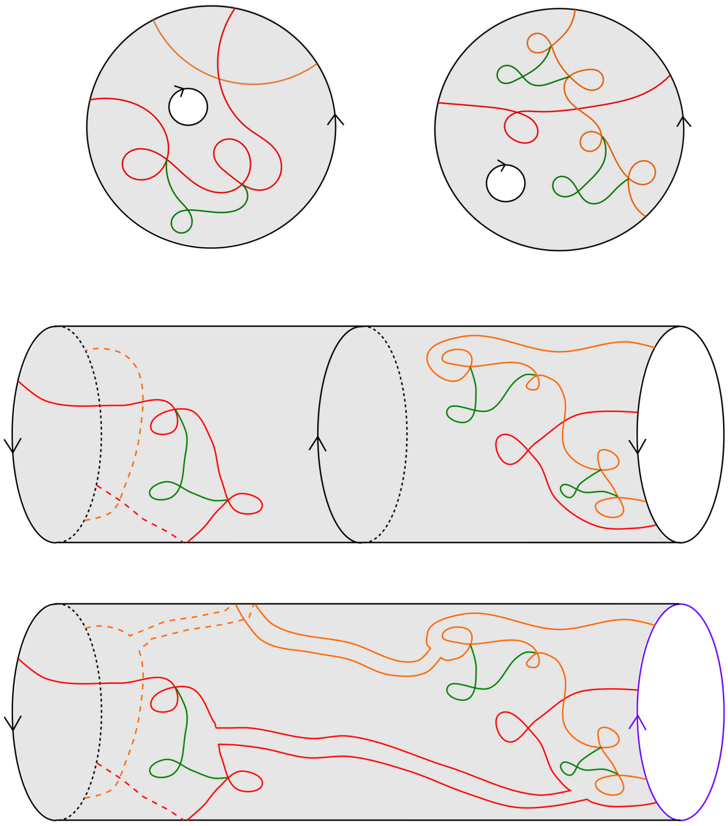

Figure 2.8: How to glue Whitney towers and get a Whitney concordance

Observe that $L_1 \#_\beta L_2$ bounds an order n twisted Whitney tower $\mathcal{U}$ obtained by performing a boundary connected sum of two copies of D^4 along the 3-balls in S^3 used to build $L_1 \#_\beta L_2$, each copy contains the Whitney tower $\mathcal{W}_i$ on L_i and $\tau_n^\infty(\mathcal{U}) = \tau_n^\infty(\mathcal{W}_1) + \tau_n^\infty(\mathcal{W}_2)$. Analogously $L'_1 \#_{\beta'} L'_2$ bounds an order n twisted Whitney tower $\mathcal{U}'$ with $\tau_n^\infty(\mathcal{U}') = \tau_n^\infty(\mathcal{W}'_1) + \tau_n^\infty(\mathcal{W}'_2)$. We have then

$$\tau_n^\infty(\mathcal{U}) = \tau_n^\infty(\mathcal{W}_1) + \tau_n^\infty(\mathcal{W}_2) = \tau_n^\infty(\mathcal{W}'_1) + \tau_n^\infty(\mathcal{W}'_2) = \tau_n^\infty(\mathcal{U}')$$

But now $L_1 \#_\beta L_2$ and $L'_1 \#_{\beta'} L'_2$ are connected by a Whitney concordance of order $n+1$ because $S^3 \times [0,1]$ is diffeomorphic to the connected sum of two 4-balls containing respectively $\mathcal{U}$ and $\mathcal{U}'$, the connected sum is realized along 4-balls in the complement of the two towers and one of the boundary spheres in the newly created $S^3 \times [0,1]$ has opposite orientation with respect to the one it had in its bounding 4-ball, see Figure 3.6 for a schematic description of this procedure. Tubing the corresponding order 0 disks in $\mathcal{U}$ and $\mathcal{U}'$ one gets then an order n Whitney concordance $\overline{\mathcal{U}}$ with

$$\tau_n^\infty(\overline{\mathcal{U}}) = \tau_n^\infty(\mathcal{U}) - \tau_n^\infty(\mathcal{U}') = 0$$

and again by Theorem 4 of previous section this says that $\overline{\mathcal{U}}$ extends to a Whitney concordance of order $n+1$ as required.

Band sum $\#$ is clearly associative and commutative and turns W_n^∞ in a group with unit the unlink and where the inverse of L is $-L$. The fact that it is finitely generated follows from the existence of a surjective group homomorphism

$$\mathcal{T}_n^\infty \xrightarrow{R_n^\infty} \mathsf{W}_n^\infty$$

called twisted realization map in [**CST12**], this takes a twisted intersection tree $t \in \mathcal{T}_n^\infty$ and returns a link bounding a twisted Whitney tower $\mathcal{W}$ of order n with $\tau_n^\infty(\mathcal{W}) = t$. The key ingredient to build this homomorphism is to adapt the framed realization procedure described in Chapter 1 Section 1.5 to the twisted setting using a twisted version of the Bing doubling construction. □

The groups W_n^∞ describe the twisted concordance filtration in the sense that if $L \in \mathbb{W}_n^\infty$ then $L \in \mathbb{W}_{n+1}^\infty$ if and only if $[L] = 0 \in \mathsf{W}_n^\infty$. In three quarter of the cases the structure of these groups is completely understood.

THEOREM 5. *For m-links and $n \not\equiv 2 \pmod 4$*

$$\mathsf{W}_n^\infty \cong \mathbb{Z}^{mr_{n+1}(m)-r_{n+2}(m)}$$

where

$$r_n(m) = \frac{1}{n}\sum_{d|n} M(d)m^{n/d}$$

and M is the Möbius function defined as $M(d) = (-1)^{l(d)}$ if d is square-free with $l(d)$ prime factors and $M(d) = 0$ if d has repeated prime factors.

Proof. There is a commutative diagram

$$\begin{tikzcd}
\mathcal{T}_n^\infty \arrow[r, "R_n^\infty"] \arrow[dr, "\eta_n"'] & \mathsf{W}_n^\infty \arrow[d, "\mu_n"] \\
 & \mathsf{D}_n
\end{tikzcd}$$

Here η_n is the group morphism that turns trees into brackets described in Chapter 1 Section 1.7 extended to twisted trees via $\eta_n(\infty - J) = \frac{1}{2}\eta_n(J - J)$ (this division by 2 makes sense in the $\mathbb{Z}$-module D_n because all terms in $\eta_n(J-J)$ have even coefficients) and this assignement is compatible with the twisted relations of $\mathcal{T}_n^\infty$.

If L is a link with $[L] \in \mathsf{W}_n^\infty$ then it bounds a twisted Whitney tower $\mathcal{W}$ of order n by definition and a twisted extension of Theorem 3 in Chapter 1 (see [**CST14**]) says that $\mu_n(L) = 0$ for $k < n$ and $\mu_n(L) = \eta_n(\tau_n^\infty(\mathcal{W}))$. If L' is another link with $[L] = [L'] \in \mathsf{W}_n^\infty$ then $L\# - L'$ bounds a twisted Whitney tower of order $n+1$ by the group-theoretic considerations of Proposition 6 above and

$$\mu_n(L) - \mu_n(L') = \eta_n(\tau_n^\infty(L)) - \eta_n(\tau_n^\infty(L')) = \eta_n(\tau_n^\infty(L\# - L')) = 0$$

This says that $\mu_n : \mathsf{W}_n^\infty \to \mathsf{D}_n$ is a well-defined group morphism. In the end of [**CST12**] is proved that for $n \not\equiv 2 \pmod 4$ the map η_n is an isomorphism and therefore also R_n^∞ and η_n are. Finally, Orr [**Orr89**] shows that D_n is free of rank $mr_{n+1} - r_{n+2}$ and this proves the statement. □

2.5 Higher Arf obstructions

The goal of this section is to describe the conjectural structure of W_n^∞ for $n \equiv 2 \pmod 4$. In this case the commutative diagram of Theorem 5 in the previous section still exists but η_n is no longer injective, the first step is then to understand its kernel.

PROPOSITION 7. *For m-links and $n = 4k-2$ we have*

$$\mathrm{Ker}(\eta_{4k-2}) \cong \mathbb{Z}_2^{r_k(m)}$$

Proof. The domain group of $\eta_{4k-2} : \mathcal{T}_{4k-2}^\infty \to \mathsf{D}_{4k-2}$ has obvious elements of 2-torsion $(J,J)^\infty$ with J rooted tree of order $k-1$, in fact $2(J,J)^\infty = 0$ because of interior twist and IHX relation in $\mathcal{T}_{4k-2}^\infty$, in fact

$$2(J,J)^\infty = \langle (J,J),(J,J) \rangle = \langle -\!\!<^{J}_{J} \, , -\!\!<^{J}_{J} \rangle = {}^{J}_{J}\!>\!\!-\!\!<^{J}_{J} = 0$$

Therefore $\langle (J,J)^\infty \rangle \subseteq \mathrm{Ker}(\eta_{4k-2})$ because D_{4k-2} is free and we claim that actually they are equal and furthermore $\langle (J,J)^\infty \rangle \cong \mathbb{Z}_2 \otimes \mathsf{L}_k$ via $(J,J)^\infty \mapsto 1 \otimes J$ (we identify here rooted trees with commutators as usual). This will prove the statement because it's known that $\mathsf{L}_k \cong \mathbb{Z}^{r_k(m)}$, see [**Reu93**] for a discussion of basis of free Lie algebras and their connection to the combinatorics of Lyndon words.

The claim is equivalent to say that in the following diagram the top row is a short exact sequence and the vertical morphism on the left is an isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & \langle (J,J)^\infty \rangle \arrow[r] \arrow[d, "(J,J)^\infty \mapsto 1 \otimes J"'] & \mathcal{T}_{4k-2}^\infty \arrow[r, "\eta_{4k-2}"] \arrow[d, "\eta_{4k-2}^\infty"] & \mathsf{D}_{4k-2} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathbb{Z}_2 \otimes \mathsf{L}_k \arrow[r, "\mathrm{sq}_{4k-2}^\infty"] & \mathsf{D}_{4k-2}^\infty \arrow[r, "p_{4k-2}"] & \mathsf{D}_{4k-2} \arrow[r] & 0
\end{tikzcd}$$

In this diagram a new group D_{4k-2}^∞ appears and also the maps in or out of it are new. The idea is to modify the target group of η_{4k-2} and lift it to a map η_{4k-2}^∞ that is an isomorphism. The new target group is defined as a pullback of abelian groups.

$$\begin{tikzcd}
\mathsf{D}_{4k-2}^\infty \arrow[r, "p_{4k-2}"] \arrow[d] & \mathsf{D}_{4k-2} \arrow[d, "\mathrm{sl}_{4k-2}"] \\
\mathbb{Z}_2 \otimes \mathsf{L}_{2k}' \arrow[r] & \mathbb{Z}_2 \otimes \mathsf{L}_{2k}
\end{tikzcd}$$

In this diagram L'_{2k} is the degree $2k$ part of the free quasi-Lie algebra on m generators. This differs from the usual free Lie algebra in that the relation $[x,x]=0$ is replaced by $[x,y]=-[y,x]$. We point out that this is a relevant modification because we are working over the ground ring $\mathbb{Z}$ and thus the two are not equivalent. In fact the first relation implies the second, this guarantees that the identity map induces a well-defined morphism of groups $\mathsf{L}'_{2k} \to \mathsf{L}_{2k}$ and we use this to define the lower horizontal map in the previous square.

The map sl_{4k-2} is way more difficult to describe, in fact it comes from the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \mathsf{D}'_{4k-2} \arrow[r] \arrow[d] & \mathsf{D}_{4k-2} \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathsf{L}'_1 \otimes \mathsf{L}'_{4k-1} \arrow[r] \arrow[d, "{[\cdot,\cdot]}"] & \mathsf{L}_1 \otimes \mathsf{L}_{4k-1} \arrow[r] \arrow[d, "{[\cdot,\cdot]}"] & 0 \\
0 \arrow[r] & \mathbb{Z}_2 \otimes \mathsf{L}_{2k} \arrow[r, "\mathrm{sq}_{4k}"] \arrow[d, "\mathrm{id}"] & \mathsf{L}'_{4k} \arrow[r] \arrow[d] & \mathsf{L}_{4k} \arrow[r] \arrow[d] & 0 \\
 & \mathbb{Z}_2 \otimes \mathsf{L}_{2k} \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}$$

Here $\mathrm{sq}_{4k}(1 \otimes x) = [x,x]$ is not the zero map because the target is a quasi-Lie algebra and it completes the natural map $\mathsf{L}'_{4k} \to \mathsf{L}_{4k}$ to a short exact sequence according to the work of Levine on free quasi-Lie algebras [**Lev06**]. The short sequence above it is also exact because it's known that the natural map $\mathsf{L}'_n \to \mathsf{L}_n$ is an isomorphism for n odd, again by Levine's work. The first and last rows are respectively kernels and cokernels and the snake lemma connect them in an exact sequence through a surjective morphism $\mathrm{sl}_{4k-2} : \mathsf{D}_{4k-2} \to \mathbb{Z}_2 \otimes \mathsf{L}_{2k}$ as required.

The map $\mathrm{sq}^{\infty}_{4k-2}$ in the first diagram is uniquely determined by the condition that it completes the first row of the following diagram to a short exact sequence and the universal property of the pullback square on the right.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2 \otimes \mathsf{L}_k \arrow[r, "\mathrm{sq}^{\infty}_{4k-2}"] \arrow[d, "\mathrm{id}"] & \mathsf{D}^{\infty}_{4k-2} \arrow[r, "p_{4k-2}"] \arrow[d] & \mathsf{D}_{4k-2} \arrow[r] \arrow[d, "\mathrm{sl}_{4k-2}"] & 0 \\
0 \arrow[r] & \mathbb{Z}_2 \otimes \mathsf{L}_k \arrow[r, "\mathrm{sq}_{2k}"] & \mathbb{Z}_2 \otimes \mathsf{L}'_{2k} \arrow[r] & \mathbb{Z}_2 \otimes \mathsf{L}_{2k} \arrow[r] & 0
\end{tikzcd}$$

The proof ends by looking at the first diagram. The map $(J,J)^{\infty} \mapsto 1 \otimes J$ is an isomorphism by direct calculation and this also implies that the first row is exact using the fact the second row is exact by the construction above. $\square$

Now being η_{4k-2} not injective the maps R^{∞}_{4k-2} and μ_{4k-2} could have nontrivial kernel. What we know for sure is that there is a short exact sequence

$$0 \to \mathrm{Ker}(\mu_{4k-2}) \to \mathsf{W}^{\infty}_{4k-2} \to \mathsf{D}_{4k-2} \to 0$$

and it splits because D_{4k-2} is free, thus

$$\mathsf{W}^{\infty}_{4k-2} \cong \mathsf{D}_{4k-2} \oplus \mathrm{Ker}(\mu_{4k-2})$$

The following conjecture was formulated by Conant, Schneiderman and Teichner in [**CST11**]

CONJECTURE 1. *The map $R^{\infty}_{4k-2} : \mathcal{T}^{\infty}_{4k-2} \to \mathsf{W}^{\infty}_{4k-2}$ is an isomorphism, consequently $\mathrm{Ker}(\mu_{4k-2}) \cong \mathrm{Ker}(\eta_{4k-2})$ and by Theorem 5 and Proposition 7 above*

$$\mathsf{W}^{\infty}_{4k-2} \cong \mathbb{Z}^{mr_{4k-1}(m)-r_{4k}(m)} \oplus \mathbb{Z}_2^{r_k(m)}$$

Here is a table of how the groups $\mathsf{W}^{\infty}_n(m)$ should appear for low n (rows) and m (columns) in this conjectural picture

	1	2	3	4	5
0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^6$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{15}$
1	0	0	$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}^{10}$
2	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}^{20} \oplus \mathbb{Z}_2^4$	$\mathbb{Z}^{50} \oplus \mathbb{Z}_2^5$
3	0	0	$\mathbb{Z}^6$	$\mathbb{Z}^{36}$	$\mathbb{Z}^{126}$
4	0	$\mathbb{Z}^3$	$\mathbb{Z}^{28}$	$\mathbb{Z}^{146}$	$\mathbb{Z}^{540}$
5	0	0	$\mathbb{Z}^{36}$	$\mathbb{Z}^{340}$	$\mathbb{Z}^{1740}$
6	0	$\mathbb{Z}^6 \oplus \mathbb{Z}_2$	$\mathbb{Z}^{126} \oplus \mathbb{Z}_2^3$	$\mathbb{Z}^{1200} \oplus \mathbb{Z}_2^6$	$\mathbb{Z}^{7050} \oplus \mathbb{Z}_2^{10}$

The first row of this table can be read as the fact that the only obstructions to rise an order 0 twisted Whitney tower on a framed m-link are the framings of its components and the pairwise linking numbers, a total of

$$m + \binom{m}{2} = \frac{m(m+1)}{2}$$

integers. The second row tells us that there are $\binom{m}{3}$ integer obstructions to rise an order 1 twisted Whitney tower, these were previously known in literature as triple linking numbers.

Coherently with Proposition 5, there is no obstruction for knots to build an order 2 twisted Whitney tower, but to rise it the classical Arf invariant of knots must vanish.

In general, it can be shown that for m-links the torsion factors correspond to the classical Arf invariant of each component and this verifies the conjecture for $4k-2=2$. This suggests the existence of a new class of concordance invariants Arf_k with values in $\mathbb{Z}_2^{r_k(m)}$. These invariants should describe the obstruction to rise a twisted Whitney tower of order $4k-2$.

DEFINITION 7. *For $k \geq 1$ the higher-order Arf obstructions are defined as*

$$\mathrm{Arf}_k = (\overline{R}^{\infty}_{4k-2})^{-1} : \mathrm{Ker}(\mu_{4k-2}) \to \frac{\mathrm{Ker}(\eta_{4k-2})}{\mathrm{Ker}(R^{\infty}_{4k-2})}$$

where $\overline{R}^{\infty}_{4k-2}$ is the quotient map of $R^{\infty}_{4k-2} : \mathrm{Ker}(\eta_{4k-2}) \to \mathrm{Ker}(\mu_{4k-2})$, well-defined and surjective by commutativity of the triangle in Theorem 5.

Observe that conjecturally Arf_k should be simply the inverse of R^{∞}_{4k-2} restricted to $\mathrm{Ker}(\mu_{4k-2})$, but the definition above is more complicated because there is no proof yet that R^{∞}_{4k-2} is injective. We emphasize that also in the conjectural picture Arf_k is not defined for all links, but only for links with suitable vanishing Milnor invariants.

CHAPTER 3

Quantum concordance invariants from $\mathfrak{sl}_2$

3.1 String links

A string link $L \subset D^2 \times [0,1]$ is the image of a smooth proper embedding $\phi : \sqcup[0,1] \hookrightarrow D^2 \times [0,1]$ such that, called ϕ_i its restriction to the ith interval, we have $\phi_i(0) = p_i \times \{0\}$ and $\phi_i(1) = p_i \times \{1\}$ for every i, here $p_i \in D^2$ are fixed interior points and $1 \leq i \leq m$ where m is the number of components L_i of the string link. We are interested in distinguishing them as embeddings up to ambient isotopies of $D^2 \times [0,1]$ fixing the endpoints, i.e. we consider $L, L' \subset D^2 \times [0,1]$ equal if there is a smooth map $H : D^2 \times [0,1] \times [0,1] \to D^2 \times [0,1]$ such that $H(\cdot,\cdot,0) = \mathrm{id}_{D^2\times[0,1]}$, $H(\cdot,\cdot,1)(L) = L'$ and $H(\cdot,\cdot,t)$ is a diffeomorphism of $D^2 \times [0,1]$ that fixes the points $p_i \times \{0\}$ and $p_i \times \{1\}$ for every i and t. A string link is trivial if it is equal in the previous sense to the string link given by $t_i \mapsto (p_i, t_i)$ on every component.

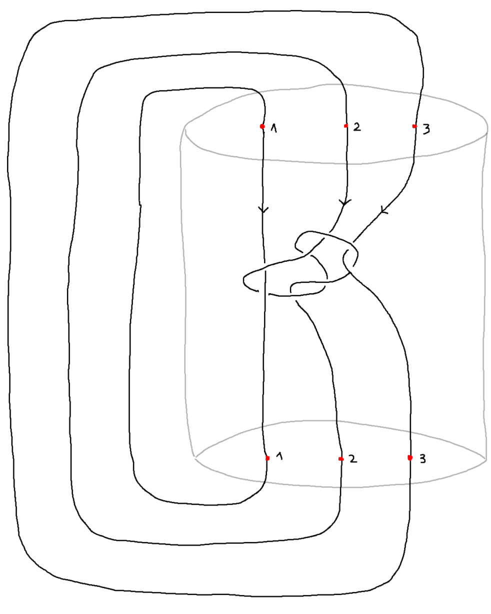

Figure 3.1: Closure

Given a string link L we can form its closure $\mathrm{cl}(L) \subset S^3$, this is a link as defined in Chapter 1. See Figure 3.1 for a description of this operation. Conversely, one can also open a link in order to get a string link as follows.

PROPOSITION 8. *Every link in S^3 is the closure of a string link in $D^2 \times [0,1]$.*

Proof. Call $D \subset \mathbb{R}^2$ a planar diagram of the link, then $\mathbb{R}^2 \setminus D$ is union of exactly one unbounded connected component E together with finitely many bounded connected components homeomorphic to open disks. Pick a point a in one of these and another point $b \in E$, if we show that there exists an embedded path $\gamma \subset \mathbb{R}^2$ connecting a and b and intersecting each link component in D exactly once then $D \setminus \gamma$ will be a planar diagram of a string link whose closure is the original link. The example in Figure 3.2 shows how starting from a one can get trapped in one of the bounded regions without any way to avoid to intersect a previously crossed component.

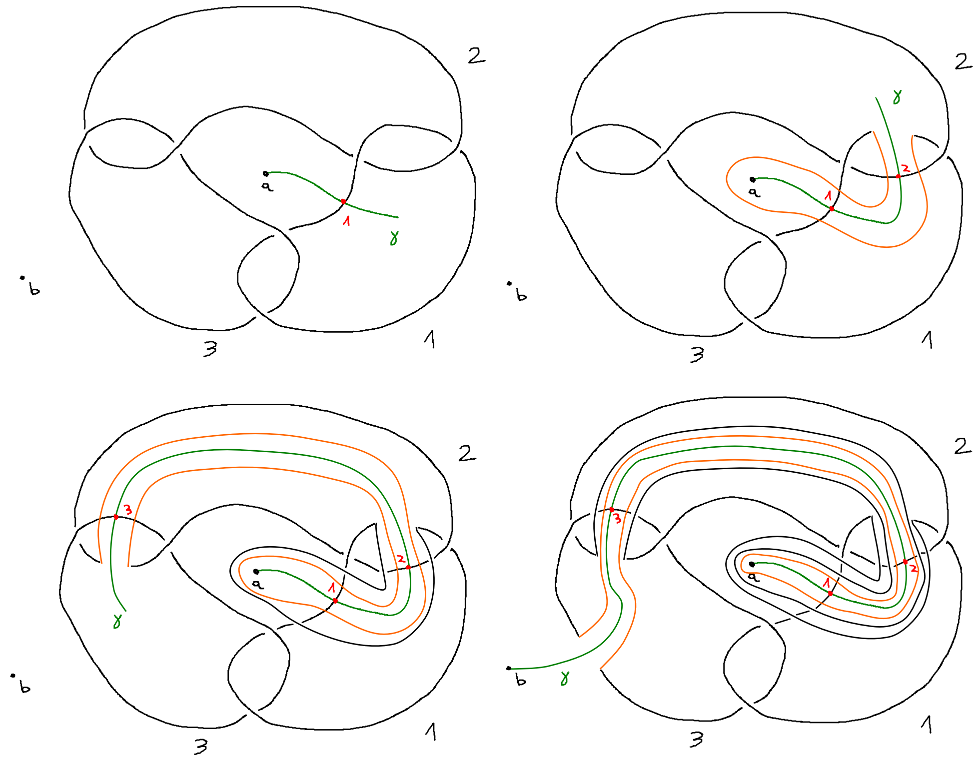

Figure 3.2: Sliding

Anyway, one can remove such a repeated intersection by sliding the obstructing component along γ up to the point a, passing over any other part of the diagram if needed. This modifies D into a new diagram D' through finitely many Reidemeister moves and thus D' represents a link isotopic to the original one in S^3 thanks to Reidemeister's theorem. Proceeding in this way we eventually get the required path γ. $\square$

In what follows, we will consider oriented framed string links, they are proper embeddings $\phi : \sqcup([0,1] \times [0,1]) \hookrightarrow D^2 \times [0,1]$ such that $\phi_i(0,-) = l_i \times \{0\}$ and $\phi_i(1,-) = l_i \times \{1\}$ for every i, here $l_i \subset D^2$ are fixed segments disjointly embedded in the interior of D^2. The closure of an oriented framed string link is analogous to the unframed one and gives a framed link in S^3.

We distinguish oriented framed string links up to isotopies preserving the framings, Figure

3.3 below shows an example of string links that are isotopic but not isotopic as framed string links.

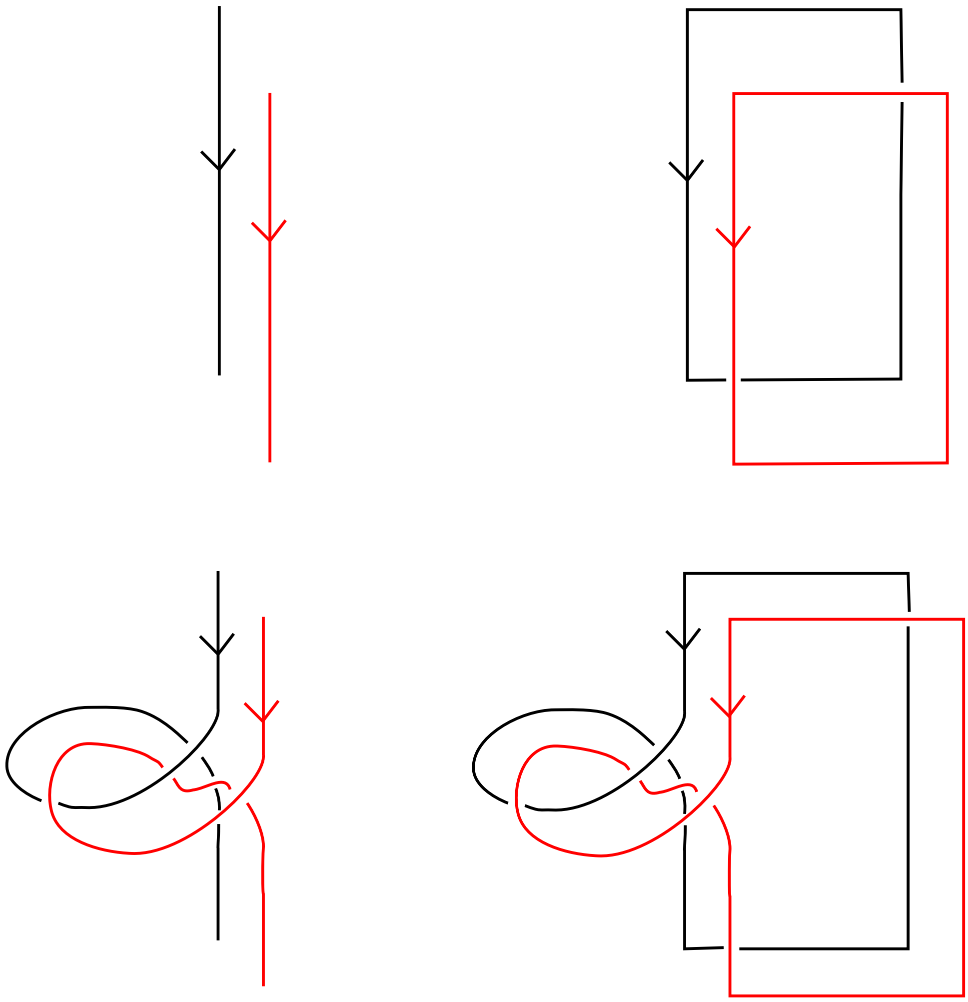

Figure 3.3: Framed string links

A planar diagram for a string link specifies a framing given by an outward vector field normal to the plane. If not specified, we will assume that a diagram describes a framed string link with this conventional framing.

3.2 Hopf algebras

In this section we introduce a class of algebras that will be used later to construct invariants of string links.

DEFINITION 8. *Fixed κ commutative ring with unit, a κ-algebra A with unit is called bialgebra if it has a compatible coalgebra structure, this means it is endowed with the following morphisms of κ-algebras*

- $\Delta : A \to A \otimes A$ *called comultiplication, the multiplication of A is denoted* $m : A \otimes A \to A$

- *$\epsilon : A \to \kappa$ called counit, the unit of A is denoted $i : \kappa \to A$ and $i(1) = 1_A$*

We require that they make the following diagrams commutative

$$\begin{tikzcd}
 & A \otimes A \arrow[dr, "\mathrm{id}_A \otimes \epsilon"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\mathrm{id}_A"'] \arrow[dr, "\Delta"'] & & A \\
 & A \otimes A \arrow[ur, "\epsilon \otimes \mathrm{id}_A"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[dr, "\mathrm{id}_A \otimes \Delta"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & A \otimes A \otimes A \\
 & A \otimes A \arrow[ur, "\Delta \otimes \mathrm{id}_A"'] &
\end{tikzcd}$$

A bialgebra is Hopf if it has a κ-linear map $S : A \to A$ called antipode such that the following diagram commutes

$$\begin{tikzcd}
 & A \otimes A \arrow[rr, "\mathrm{id}_A \otimes S"'] & & A \otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \kappa \arrow[rr, "i"] & & A \\
 & A \otimes A \arrow[rr, "S \otimes \mathrm{id}_A"] & & A \otimes A \arrow[ur, "m"'] &
\end{tikzcd}$$

We are interested in a particular class of Hopf algebras. We say that A is quasi-triangular if it has a universal R-matrix, i.e. an element $R \in A \otimes A$ such that called $R = \sum \alpha_i \otimes \beta_i$ one has

- R invertible
- $(P \circ \Delta)(x) = R\Delta(x)R^{-1}$ where $P(a_1 \otimes a_2) = a_2 \otimes a_1$
- $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$ where $R_{23} = 1 \otimes R$ and $R_{13} = \sum \alpha_i \otimes 1 \otimes \beta_i$
- $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$ where $R_{12} = R \otimes 1$

SWEEDLER'S NOTATION

For $x \in A$ we write the element $\Delta(x) = \sum_i x_i' \otimes x_i'' \in A \otimes A$ as the sum

$$\Delta(x) = \sum_{(x)} x' \otimes x''$$

The coassociativity of A gives in this notation the following identity

$$\sum_{(x)} \left(\sum_{(x')} (x')' \otimes (x')'' \right) \otimes x'' = \sum_{(x)} x' \otimes \left(\sum_{(x'')} (x'')' \otimes (x'')'' \right)$$

and we therefore write simply

$$\Delta^{(2)}(x) = \sum_{(x)} x' \otimes x'' \otimes x'''$$

The counitality of A reads instead

$$\sum_{(x)} \epsilon(x')x'' = x = \sum_{(x)} x'\epsilon(x'')$$

If A, B are bialgebras the set of κ-linear maps $\mathrm{Hom}_\kappa(A, B)$ is a group with respect to the convolution operation defined as

$$f * g = m_B \circ (f \otimes g) \circ \Delta_A$$

this reads in Sweedler's notation as

$$(f * g)(x) = \sum_{(x)} f(x')g(x'')$$

The unit in this group is $i_B\epsilon_A$ The following proposition summarizes the key properties of quasi-triangular Hopf algebras that will be used later.

PROPOSITION 9. *Given a quasi-triangular Hopf algebra A with R-matrix R and antipode S*

1. $S(xy) = S(y)S(x)$ *for all* $x, y \in A$
2. $R^{-1} = (S \otimes \mathrm{id})(R) = (\mathrm{id} \otimes S^{-1})(R)$
3. $(S \otimes S)(R) = R$
4. $S^2(x) = uxu^{-1}$ *for all* $x \in A$
5. $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ *(Yang-Baxter equation)*

Proof. (1) Define $\nu, \rho \in \mathrm{Hom}_\kappa(A \otimes A, A)$ as $\nu(x \otimes y) = S(y)S(x)$ and $\rho(x \otimes y) = S(xy)$, then the claim is equivalent to $\nu = \rho$. Being $\mathrm{Hom}_\kappa(A \otimes A, A)$ a group with respect to convolution, this is equivalent to prove $\rho * m = m * \nu = i_A\epsilon_{A\otimes A}$

$$\begin{aligned}
(\rho * m)(x \otimes y) &= \sum_{(x\otimes y)} \rho((x \otimes y)')m((x \otimes y)'') \\
&= \sum_{(x),(y)} \rho(x' \otimes y')m(x'' \otimes y'') \\
&= \sum_{(x),(y)} S(x'y')x''y'' \\
&= \sum_{(xy)} S((xy)')(xy)'' = i\epsilon(x \otimes y)
\end{aligned}$$

In the computation above, we used that the coproduct of $A \otimes A$ is given by $\Delta(x \otimes y) = (\mathrm{id} \otimes P \otimes id)(\Delta \otimes \Delta)(x \otimes y)$ to get from the first to the second line, the definition of ρ to get from the second to the third line, the fact that $\Delta(ab) = \Delta(a)\Delta(b)$ given by the bialgebra structure of A to get from third to the fourth line, the last equality uses $S * \mathrm{id} = i\epsilon$ and this is just the hexagon relation that holds in any Hopf algebra in a compact form. Similar arguments justify the following computation

$$\begin{aligned}(m * \nu)(x \otimes y) &= \sum_{(x \otimes y)} m((x \otimes y)')\nu((x \otimes y)'') \\ &= \sum_{(x),(y)} x'y'S(y'')S(x'') \\ &= \sum_{(x)} x' \left(\sum_{(y)} y'S(y'') \right) S(x'') \\ &= \sum_{(x)} x' i\epsilon(y)S(x'') = i\epsilon(x)i\epsilon(y) = i\epsilon(xy)\end{aligned}$$

(2) We start by observing that $(\epsilon \otimes \mathrm{id})(R) = 1 = (\mathrm{id} \otimes \epsilon)(R)$. In fact by counit definition $(\epsilon \otimes \mathrm{id})\Delta = \mathrm{id}$, by quasi-triangular property $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$. These two facts together give

$$\begin{aligned}R &= (\mathrm{id} \otimes \mathrm{id})(R) = ((\epsilon \otimes \mathrm{id})\Delta \otimes \mathrm{id})(R) = (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) \\ &= (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(R_{13}R_{23}) = (\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(R_{13})\epsilon(1)R = (\epsilon \otimes \mathrm{id})(R)R\end{aligned}$$

and multiplying by R^{-1} on both sides we get $(\epsilon \otimes \mathrm{id})(R) = 1$, the other is analogous. Now hexagon relation of Hopf algebras says that $m(S \otimes \mathrm{id})\Delta(x) = \epsilon(x)1$ for all $x \in A$, this implies together with the previous observation that

$$(m \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) = (m(S \otimes \mathrm{id})\Delta \otimes \mathrm{id})(R) = (\epsilon \otimes \mathrm{id})(R) = 1$$

Using this identity we obtain

$$\begin{aligned}1 &= (m \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) = (m \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(R_{13})S(1)R \\ &= (S \otimes \mathrm{id})(R)S(1)R = (S \otimes \mathrm{id})(R)R\end{aligned}$$

The last equality depends on the fact that, using 1), we have $S(1) = S(11) = S(1)S(1)$ and being S invertible $S(1) = 1$. Multiplying the previous identity on both sides by R^{-1} we get $(S \otimes \mathrm{id})(R) = R^{-1}$ and a similar argument shows that $(\mathrm{id} \otimes S^{-1})(R) = R^{-1}$.

(3) Using the previous point we get

$$\begin{aligned}(S \otimes S)(R) &= (\mathrm{id} \otimes S)(S \otimes \mathrm{id})(R) = (\mathrm{id} \otimes S)(R^{-1}) \\ &= (\mathrm{id} \otimes S)(\mathrm{id} \otimes S^{-1})(R) = (\mathrm{id} \otimes \mathrm{id})(R) = R\end{aligned}$$

(4) Quasi-triangular property of A gives $P\Delta(x) = R\Delta(x)R^{-1}$ for all $x \in A$. Then we have for all $y \in A \otimes A$ with $y = y_1 \otimes y_2$ that

$$\begin{aligned}(P\Delta \otimes \mathrm{id})(y)(R \otimes 1) &= (R\Delta(y_1)R^{-1} \otimes y_2)(R \otimes 1) \\ &= R\Delta(y_1) \otimes y_2 = (R \otimes 1)(\Delta \otimes \mathrm{id})(y)\end{aligned}$$

Applying now this identity to $y = \Delta(x) = \sum_{(x)} x' \otimes x''$ we get that the following identity $[A]$

$$(P\Delta \otimes \mathrm{id}) \left(\sum_{(x)} x' \otimes x'' \right) \left(\sum_i \alpha_i \otimes \beta_i \otimes 1 \right) =$$

$$= \left(\sum_{(x)} x'' \otimes x' \otimes x''' \right) \left(\sum_i \alpha_i \otimes \beta_i \otimes 1 \right) =$$

$$= \sum_{(x),i} x''\alpha_i \otimes x'\beta_i \otimes x'''$$

must agree with the following identity $[B]$

$$(R \otimes 1)(\Delta \otimes \mathrm{id}) \left(\sum_{(x)} x' \otimes x'' \right) = \left(\sum_i \alpha_i \otimes \beta_i \otimes 1 \right) \left(\sum_{(x)} x' \otimes x'' \otimes x''' \right) =$$

$$= \sum_{(x),i} \alpha_i x' \otimes \beta_i x'' \otimes x'''$$

Putting $[A] = [B]$ and applying to both sides the map $A \otimes A \otimes A \to A$ that applies $\mathrm{id} \otimes S \otimes S^2$ and then multiplies tensors from right to left we get

$$\sum_{(x),i} S^2(x''')S(x'\beta_i)x''\alpha_i = \sum_{(x),i} S^2(x''')S(\beta_i x'')\alpha_i x'$$

and using 1) we obtain

$$\sum_{(x),i} S^2(x''')S(\beta_i)S(x')x''\alpha_i = \sum_{(x),i} S^2(x''')S(x'')S(\beta_i)\alpha_i x'$$

We now prove that LHS=$S^2(x)u$ and RHS=ux and thus the claim. For the left hand side

$$\sum_{(x)} S(x')x'' \otimes x''' = \sum_{(x)} \epsilon(x')1 \otimes x'' = 1 \otimes \left(\sum_{(x)} \epsilon(x') \otimes x'' \right) = 1 \otimes x$$

where we used the hexagon relation $S * \mathrm{id} = i\epsilon$ for the first equality and the counit property $(\epsilon \otimes \mathrm{id})\Delta = \mathrm{id}$ for the last equality. Applying $\mathrm{id} \otimes S^2$ to the previous identity we get

$$\sum_{(x)} S(x')x'' \otimes S^2(x''') = 1 \otimes S^2(x)$$

and multiplying both sides on the right by $\sum_i \alpha_i \otimes S(\beta_i)$ we have

$$\sum_{(x),i} S(x')x''\alpha_i \otimes S^2(x'')S(\beta_i) = \sum_i \alpha_i \otimes S^2(x)S(\beta_i)$$

Now recall that by definition $u = \sum_i S(\beta_i)\alpha_i$ and therefore

$$S^2(x)u = \sum_i S^2(x)S(\beta_i)\alpha_i = \sum_{(x),i} S^2(x''')S(\beta_i)S(x')x''\alpha_i$$

the last equality follows by applying mP to both sides of the previous identity. For the right hand side

$$\sum_{(x)} x' \otimes S(x''S(x''')) = \sum_{(x)} x' \otimes S(\epsilon(x'')1) = \sum_{(x)} x'\epsilon(x'') \otimes S(1) = x \otimes 1$$

where the first equality uses 1) and the hexagon relation $\mathrm{id} * S = i\epsilon$, the last equality uses the counit property $(\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}$ and the already mentioned fact that $S(1) = 1$. Multiplying on the left both sides of the last identity by $u \otimes 1$ we get

$$\sum_{(x)} ux' \otimes S^2(x''')S(x'') = ux \otimes 1$$

and by definition of u this is equivalent to

$$\sum_{(x),i} S(\beta_i)\alpha_i x' \otimes S^2(x''')S(x'') = ux \otimes 1$$

Applying now mP to both sides we finally get

$$ux = \sum_{(x),i} S^2(x''')S(x'')S(\beta_i)\alpha_i x'$$

(5) Using the properties of quasi-triangular algebra $R_{13}R_{23} = (\Delta \otimes \mathrm{id})(R)$ and $R\Delta(x) = P\Delta(x)R$ we have

$$\begin{aligned}
R_{12}R_{13}R_{23} &= (R \otimes 1)(\Delta \otimes \mathrm{id})(R) = \sum_i (R \otimes 1)(\Delta(\alpha_i) \otimes \beta_i) \\
&= \sum_i R\Delta(\alpha_i) \otimes \beta_i = \sum_i P\Delta(\alpha_i)R \otimes \beta_i \\
&= (P \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R)(R \otimes 1) = (P \otimes \mathrm{id})(R_{13}R_{23})(R \otimes 1) = R_{23}R_{13}R_{12}
\end{aligned}$$

The last equality follows from the definition of R_{12} and the following computation

$$\begin{aligned}
(P \otimes \mathrm{id})(R_{13}R_{23}) &= (P \otimes \mathrm{id})\left(\sum_i \alpha_i \otimes 1 \otimes \beta_i\right)\left(\sum_j 1 \otimes \alpha_j \otimes \beta_j\right) \\
&= (P \otimes \mathrm{id})\left(\sum_{i,j} \alpha_i \otimes \alpha_j \otimes \beta_i\beta_j\right) = \sum_{i,j} \alpha_j \otimes \alpha_i \otimes \beta_i\beta_j \\
&= \left(\sum_i 1 \otimes \alpha_i \otimes \beta_i\right)\left(\sum_j \alpha_j \otimes 1 \otimes \beta_j\right) = R_{23}R_{13}
\end{aligned}$$

□

A quasi-triangular Hopf algebra (A, R) is ribbon if called $u = \sum S(\beta_i)\alpha_i$ there is a special square root of $S(u)u$, more precisely if there exists $v \in A$ such that

- $v^2 = S(u)u$
- v is central in A
- $\Delta(v) = (v \otimes v)(R_{21}R)^{-1}$ where $R_{21} = \sum \beta_i \otimes \alpha_i$
- $S(v) = v$
- $\epsilon(v) = 1$

3.3 Universal quantum invariants

We decribe now the construction of an invariant of framed string links with m components $\mathcal{Q}^A$ with values in $A^{\otimes m}$. This is an adaptation to string links of the original definition of universal quantum A-invariant due to Lawrence [**Law89, Law90**].

Given an oriented string link with m components $L \subset D^2 \times [0, 1]$ it admits a sliced diagram, i.e. a planar diagram made up of stacked slices, each one containing finitely many elementary tangles (see Figure 3.4) and with at most one of them different from a vertical straight line.

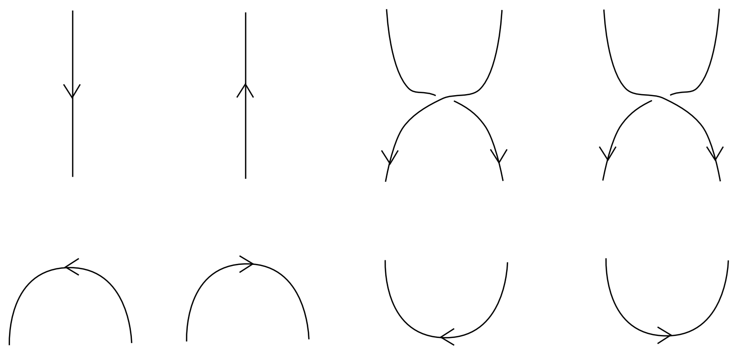

Figure 3.4: Elementary tangles

A theorem of Turaev [**Tur90**] analogous to Reidmeister's theorem for links says that two sliced diagrams describe the same framed string link if and only if they differ by finitely many moves in the set of Figure 3.5

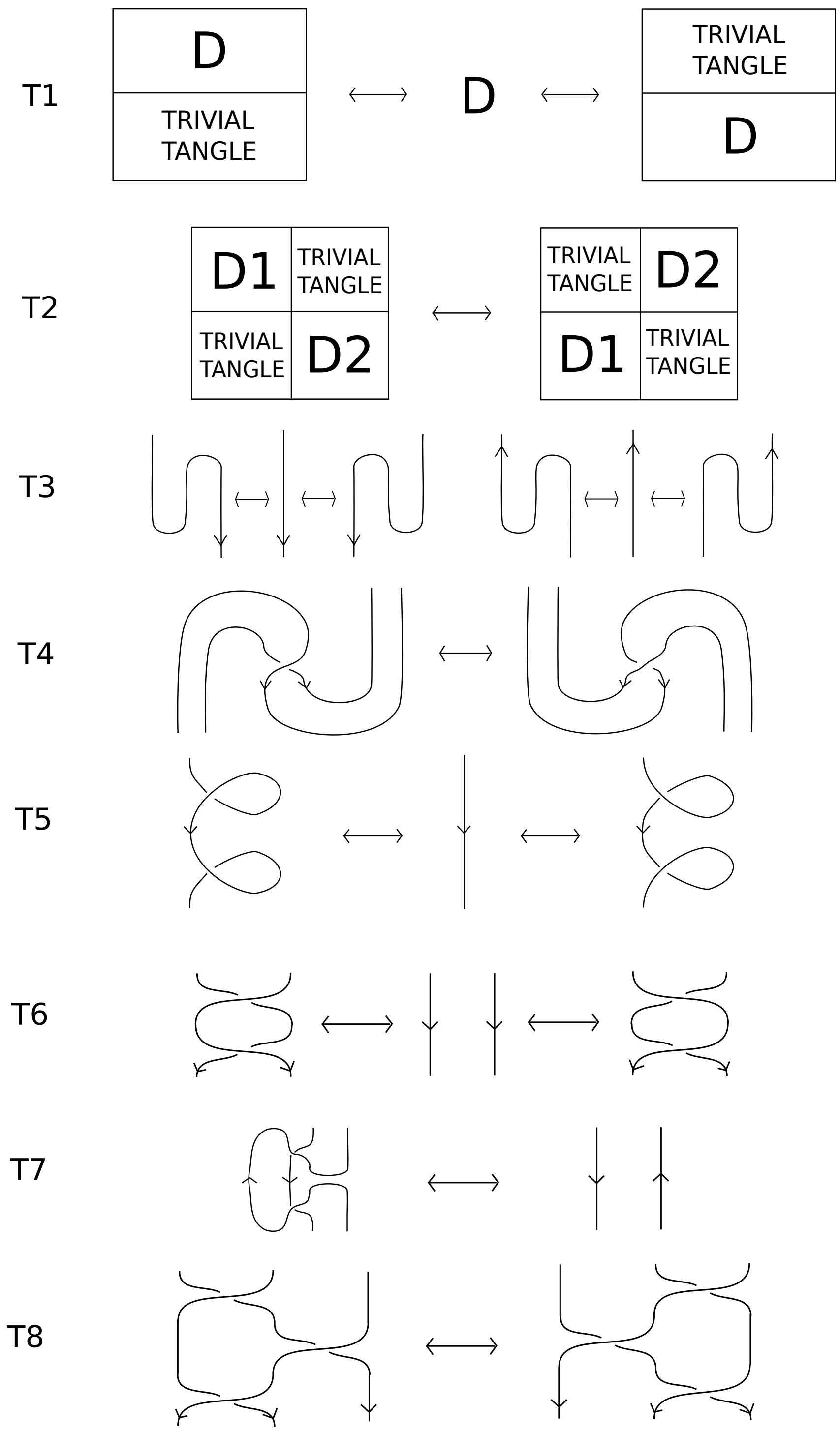

Figure 3.5: Turaev moves

We define $\mathcal{Q}^A(L)$ by giving its values on elementary tangles as in Figure 3.6, then computing the product in A of elements labelling the marked points on a component of L in the reverse order specified by the orientation and then tensoring the results of each component to get an element in $A^{\otimes m}$.

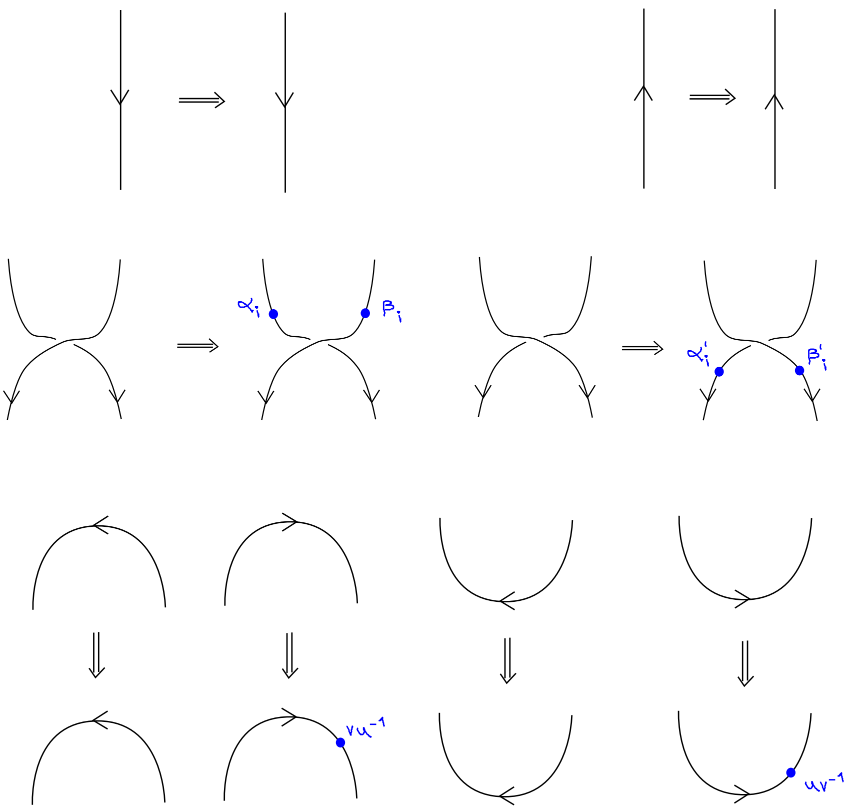

Figure 3.6: Quantum invariant

THEOREM 6. *Given $L \subset D^2 \times [0,1]$ oriented framed string link and (A, R, v) ribbon Hopf algebra, $\mathcal{Q}^A(L) \in A^{\otimes m}$ only depends on the isotopy class of L as oriented framed string link.*

Proof. By Turaev's theorem, it suffices to prove that $\mathcal{Q}^A(L)$ does not change if computed using two sliced diagrams of L differing by finitely many Turaev moves.

(T1) $\mathcal{Q}^A$ has value 1 on straight lines by definition, this guarantees that stacking trivial tangles doesn't change its value.

(T2) If D_1, D_2 are two sliced diagrams and $D_1 \otimes D_2$ is their juxtaposition then $\mathcal{Q}^A(D_1 \otimes D_2) = \mathcal{Q}^A(D_1) \otimes \mathcal{Q}^A(D_2)$ because there are no crossings between strings of different diagrams. Then $\mathcal{Q}^A$ has the same value on the two diagrams of the move because they only differ by (T1) moves on the two halves.

(T3) Invariance under this move depends on the definition of $\mathcal{Q}^A$ on maxima/minima, see Figure 3.7

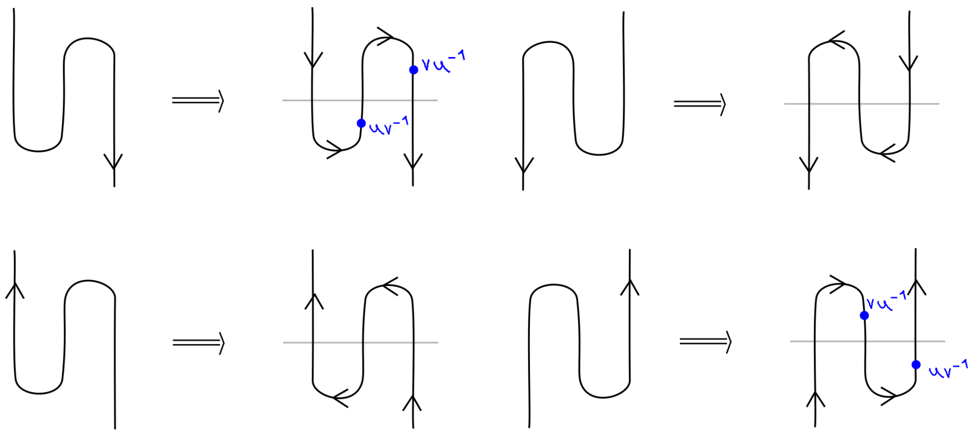

Figure 3.7: T3 invariance

(T4) The value of $\mathcal{Q}^A$ on the two diagrams involved in this move are computed as in Figure 3.8

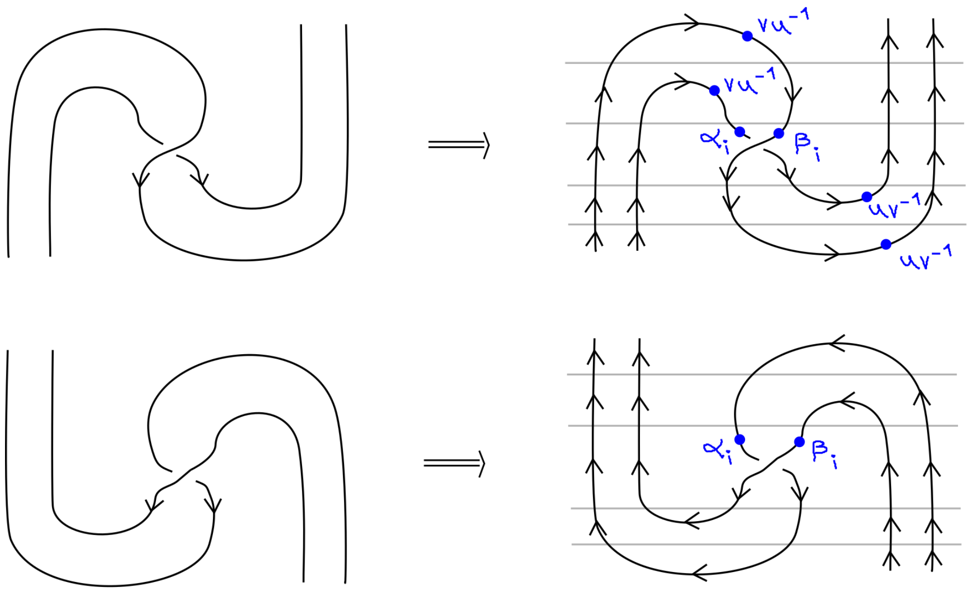

Figure 3.8: T4 invariance

To prove that they agree, first notice that $(u \otimes u)R = R(u \otimes u)$ because

$$(u \otimes u)R = \sum_i u\alpha_i \otimes u\beta_i = \sum_i S^2(\alpha_i)u \otimes S^2(\beta_i)u =$$

$$= (S^2 \otimes S^2)(R)(u \otimes u) = R(u \otimes u)$$

where we used properties 3) of the previous proposition for the second equality and 2) for the last equality. This fact, together with the fact that $v \in A$ is central by definition of ribbon algebra gives

$$\sum_i uv^{-1}\alpha_i vu^{-1} \otimes uv^{-1}\beta_i vu^{-1} = \sum_i u\alpha_i u^{-1} \otimes u\beta_i u^{-1} =$$

$$= (u \otimes u)R(u^{-1} \otimes u^{-1}) = R(u \otimes u)(u^{-1} \otimes u^{-1}) = R$$

(T5) Figure 3.9 shows how to split one of the two non-straight diagrams D in this move in two stacked subdiagrams D_1, D_2, we write this as $D = D_2 D_1$. The other non-straight diagram splits as $D' = D_1 D_2$.

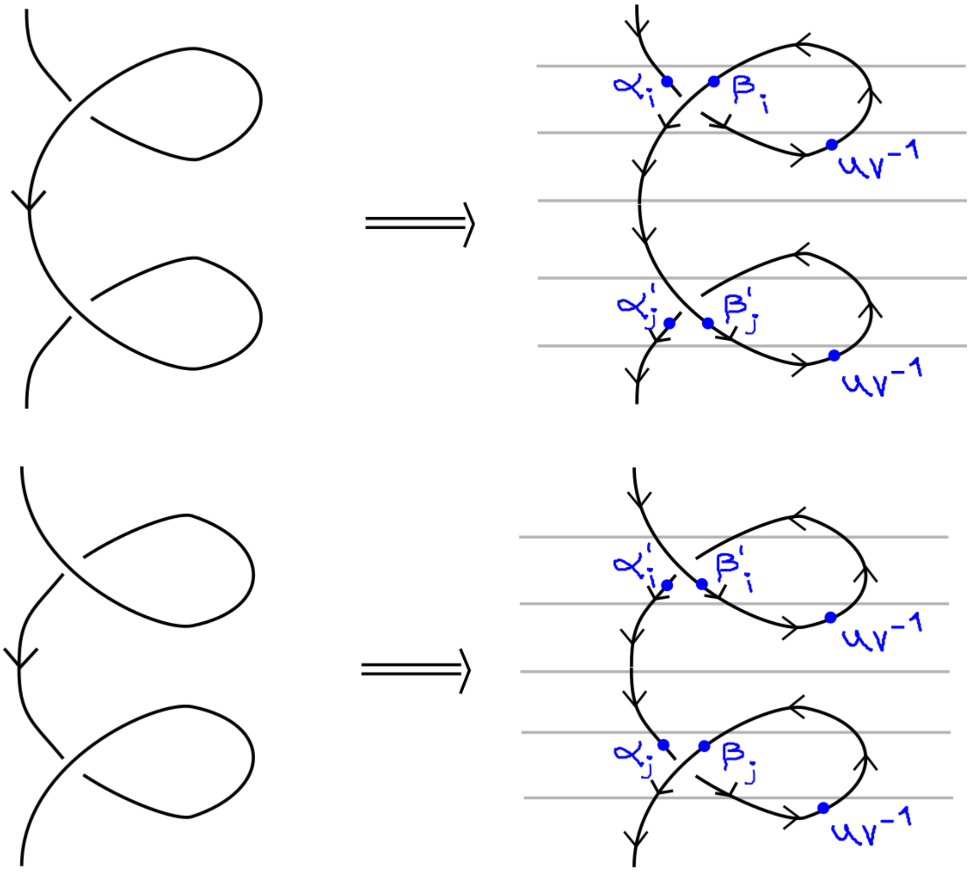

Figure 3.9: T5 invariance

Then $\mathcal{Q}^A(D_2 D_1) = \mathcal{Q}^A(D_2)\mathcal{Q}^A(D_1)$ and $\mathcal{Q}^A(D_1 D_2) = \mathcal{Q}^A(D_1)\mathcal{Q}^A(D_2)$ and we show that $\mathcal{Q}^A(D_1) = v$ and $\mathcal{Q}^A(D_2) = v^{-1}$.

$$\mathcal{Q}^A(D_2) = \sum_i \beta_i u v^{-1} \alpha_i = v^{-1} \sum_i \beta_i u \alpha_i = v^{-1} u \sum_i S^{-2}(\beta_i)\alpha_i$$

where the second equality uses that v (and thus v^{-1} is central and the third depends on property 4) in the previous proposition. Now using property 2) of the proposition we get

$$u^{-1} = \sum_i S^{-1}(\beta'_i)\alpha'_i = mP(\mathrm{id} \otimes S^{-1})R^{-1} = mP(\mathrm{id} \otimes S^{-1})(S \otimes \mathrm{id})(R) =$$

$$= mP(S \otimes S^{-1})(R) = mP(\mathrm{id} \otimes S^{-2})(S \otimes S)(R) = mP(\mathrm{id} \otimes S^{-2})(R) = \sum_i S^{-2}(\beta_i)\alpha_i$$

and this concludes the proof that $\mathcal{Q}^A(D_2) = v^{-1}uu^{-1} = v^{-1}$. Analogously we have

$$\mathcal{Q}^A(D_1) = \sum_i \alpha'_i u v^{-1} \beta'_i = v^{-1} \sum_i \alpha'_i u \beta'_i = v^{-1} u \sum_i S^{-2}(\alpha'_i)\beta'_i$$

Now using again 2) we have

$$\sum_i S^{-2}(\alpha'_i)\beta'_i = m(S^{-2} \otimes \mathrm{id})R^{-1} = m(S^{-2} \otimes \mathrm{id})(S \otimes \mathrm{id})(R) =$$

$$= m(S^{-1} \otimes \mathrm{id})(R) = \sum_i S^{-1}(\alpha_i)\beta_i$$

Finally, using a combination of 2) and 1) in the proposition we get

$$\sum_i S^{-1}(\alpha_i)\beta_i = m(S^{-1} \otimes \mathrm{id})(R) = m(S^{-1} \otimes \mathrm{id})(S^2 \otimes S^2)(R) =$$

$$= m(S \otimes S^2)(R) = \sum_i S(\alpha_i)S^2(\beta_i) = \sum_i S(S(\beta_i)\alpha_i) = S(u)$$

and this concludes the proof that $\mathcal{Q}^A(D_1) = v^{-1}uS(u) = v^{-1}v^2 = v$, here we used the fact that $v^2 = S(u)u$ by definition of ribbon algebra and that $S(u)u = uS(u)$ by property 4) in the previous proposition.

(T6) Invariance under this move depends on invertibility of the R-matrix.

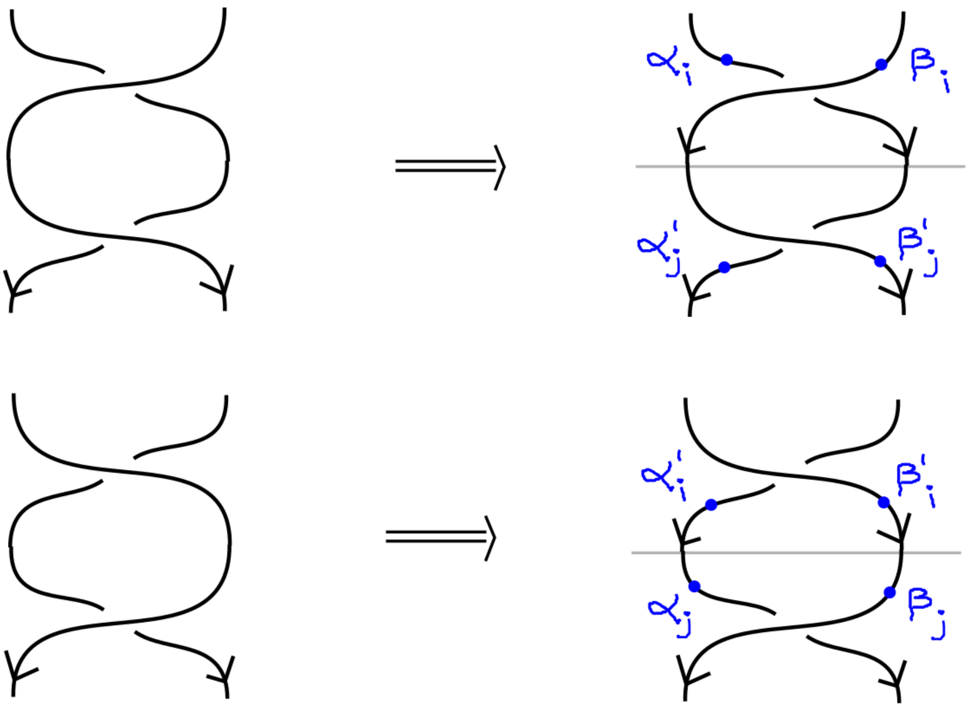

Figure 3.10: T6 invariance

The value of $\mathcal{Q}^A$ on the diagram in Figure 3.10a is

$$\sum_{i,j} \alpha'_j \alpha_i \otimes \beta'_j \beta_i = R^{-1}R = 1 \otimes 1$$

and its value on the diagram in Figure 3.10b is

$$\sum_{i,j} \beta_j \beta_i' \otimes \alpha_j \alpha_i' = P(R)P(R^{-1}) = P(RR^{-1}) = P(1 \otimes 1) = 1 \otimes 1$$

(T7) We show that the value of $\mathcal{Q}^A$ on the diagram in Figure 3.11 is $1 \otimes 1$.

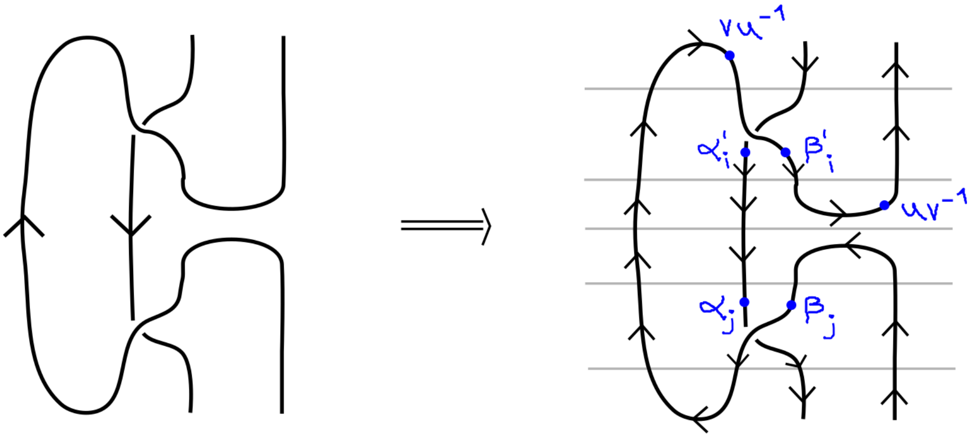

Figure 3.11: T7 invariance

Using the properties proved in the previous proposition we have

$$\sum_{i,j} \alpha_j \alpha_i' \otimes uv^{-1}\beta_i' vu^{-1}\beta_j = \sum_{i,j} \alpha_j \alpha_i' \otimes u\beta_i' u^{-1}\beta_j = \sum_{i,j} \alpha_j \alpha_i' \otimes S^2(\beta_i')\beta_j) =$$

$$= \sum_{i,j} \alpha_j \alpha_i' \otimes S(S^{-1}(\beta_j)S(\beta_i')) = (\mathrm{id} \otimes S) \left[\sum_{i,j} \alpha_j \alpha_i \otimes S^{-1}(\beta_j)S(\beta_i') \right] =$$

$$= (\mathrm{id} \otimes S) \left[(\mathrm{id} \otimes S^{-1})(R)(\mathrm{id} \otimes S)(R^{-1}) \right] = (\mathrm{id} \otimes S)(R^{-1}R) =$$

$$= (\mathrm{id} \otimes S)(1 \otimes 1) = 1 \otimes S(1) = 1 \otimes 1$$

(T8) Invariance under this move follows from the fact that the values of $\mathcal{Q}^A$ on the two diagrams equal the two sides of the Yang-Baxter equation satisfied by the R-matrix, see Figure 3.12.

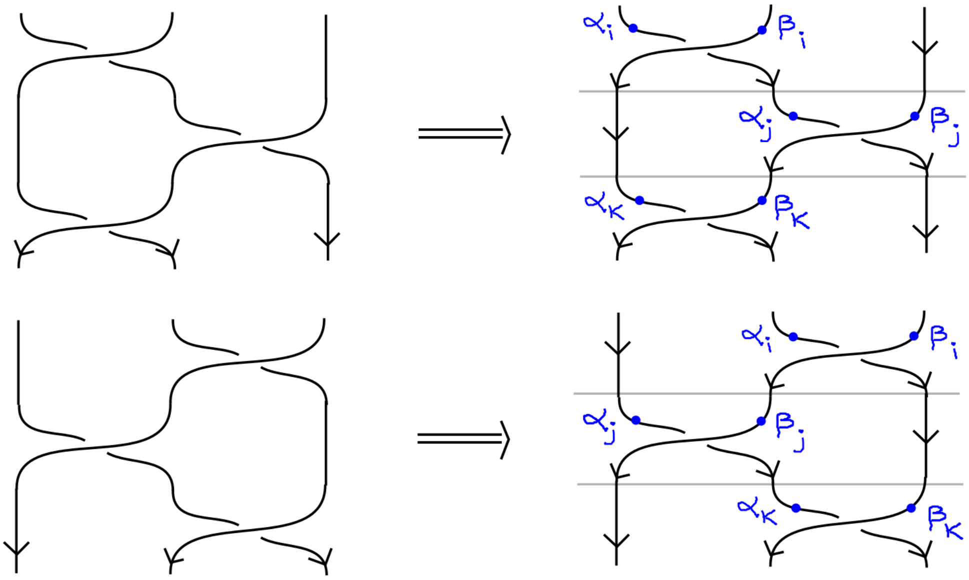

Figure 3.12: T8 invariance

In detail

$$R_{12}R_{13}R_{23} = (R\otimes 1)\left(\sum_j \alpha_j \otimes 1 \otimes \beta_j\right)(1\otimes R) = \sum_{i,j,k} \alpha_i\alpha_j \otimes \beta_i\alpha_k \otimes \beta_j\beta_k$$

and

$$R_{23}R_{13}R_{12} = (1\otimes R)\left(\sum_j \alpha_j \otimes 1 \otimes \beta_j\right)(R\otimes 1) = \sum_{i,j,k} \alpha_j\alpha_k \otimes \alpha_i\beta_k \otimes \beta_i\beta_j$$

□

3.4 Drinfeld algebra $U_h(\mathfrak{sl}_2)$

In this section we introduce an example of ribbon Hopf algebra originally due to Drinfeld and called $U_h(\mathfrak{sl}_2)$ or h-adic quantized enveloping algebra of $\mathfrak{sl}_2$, [**Dri87**] contains a description of this algebra and a more general overview of the philosophy of quantum groups. We point out that this is not what is usually called quantized enveloping algebra of $\mathfrak{sl}_2$ by many authors, for example [**Jan95**].

The construction takes place in the category of complete h-adic algebras over $\kappa = \mathbb{Q}[[h]]$. The objects of this category are topological algebras A with respect to the topology whose open sets are $x + h^nA$ for $x \in A$ and $n \in \mathbb{N}$ and such that every Cauchy sequence converges to a limit. The morphisms of this category are continuous morphisms of $\mathbb{Q}[[h]]$-algebras. All definitions and results of previous section specialize to this category by asking algebras to have the h-adic topology, maps to be continuous and taking completions.

Let $\mathbb{Q}[[h]]$ be the ring of formal power series with rational coefficients and form $\mathcal{F}$ free associative $\mathbb{Q}[[h]]$-algebra with unit on the set $\{E, F, H\}$. Let $(h) \subset \mathbb{Q}[[h]]$ be the ideal generated by h and put on $\mathcal{F}$ the h-adic topology. Define

$$q = \exp(h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} \in \mathbb{Q}[[h]]$$

$$K = \exp\left(\frac{hH}{2}\right) = \sum_{n=0}^{\infty} \frac{h^n}{n!2^n} H^n \in \hat{\mathcal{F}}$$

The h-adic quantized enveloping algebra $U_h = U_h(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is then the complete h-adic algebra obtained by taking the quotient of $\hat{\mathcal{F}}$ by the closure of the ideal generated by the following relations

$$HE = E(H+2), \quad HF = F(H-2), \quad EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}}$$

This quotient is again a topological $\mathbb{Q}[[h]]$-algebra with respect to the quotient topology and this is the same as the h-adic topology, for this technical point see Chapter I page 39 Theorem 5.21 in [**War93**]. Finally, take the completion of this.

We will now describe a structure of ribbon Hopf algebra on $U_h = U_h(\mathfrak{sl}_2)$.

PROPOSITION 10. *There exists a unique morphism of complete h-adic $\mathbb{Q}[[h]]$-algebras $\Delta : U_h \to U_h \otimes U_h$ such that*

$$\begin{aligned}
\Delta(E) &= E \otimes 1 + K \otimes E \\
\Delta(F) &= F \otimes K^{-1} + 1 \otimes F \\
\Delta(H) &= H \otimes 1 + 1 \otimes H
\end{aligned}$$

furthermore Δ satisfies the coassociativity condition.

Proof. If such a morphism exists then it's unique because $\{E, F, H\}$ generate a dense subalgebra of U_h by definition. On the other hand we can define Δ on these generators as in the statement and get a morphism of $\mathbb{Q}[[h]]$-algebras $\mathcal{F} \to U_h \otimes U_h$, then extend by continuity to a morphism $\hat{\mathcal{F}} \to U_h \otimes U_h$ of h-adic $\mathbb{Q}[[h]]$-algebras and define $\Delta : U_j \to U_h \otimes U_h$ to be the quotient morphism. For this last step we must check that Δ vanishes on the closed ideal generated by the three relations defined above.

$$\begin{aligned}
\Delta(HE - EH - 2E) &= \Delta(H)\Delta(E) - \Delta(E)\Delta(H) - 2\Delta(E) = \\
&= HE \otimes 1 + HK \otimes E + E \otimes H + K \otimes HE - EH \otimes 1 \\
&\qquad - E \otimes H - KH \otimes E - K \otimes EH - 2E \otimes 1 - 2K \otimes E = \\
&= (HK - KH) \otimes E = 0
\end{aligned}$$

The last equality holds because $HK = KH$ being K a sum of powers of H.

$$\begin{aligned}
\Delta(HF - FH + 2F) &= \Delta(H)\Delta(F) - \Delta(F)\Delta(H) + 2\Delta(F) = \\
&= HF \otimes K^{-1} + H \otimes F + F \otimes HK^{-1} + 1 \otimes HF - FH \otimes K^{-1} \\
&\qquad - F \otimes K^{-1}H - H \otimes F - 1 \otimes FH + 2F \otimes K^{-1} + 21 \otimes F = \\
&= F \otimes (HK^{-1} - K^{-1}H) = 0
\end{aligned}$$

The last equality holds because $HK^{-1} = K^{-1}H$ being K^{-1} a sum of powers of H. For the third relation, a preliminary observation is that using continuity of Δ and $\otimes$ we have

$$\Delta(K) = \Delta\left(\sum_{n=0}^{\infty} \frac{h^n}{n!2^n} H^n\right) = \sum_{n=0}^{\infty} \frac{h^n}{n!2^n} \Delta(H)^n =$$

$$= \sum_{n=0}^{\infty} \frac{h^n}{n!2^n}\left(\sum_{k=0}^{n} \binom{n}{k} H^k \otimes H^{n-k}\right) = \sum_{n,k} \frac{h^{n+k}}{n!k!2^{n+k}} H^n \otimes H^k =$$

$$= \left(\sum_{n=0}^{\infty} \frac{h^n}{n!2^n} H^n\right)\left(\sum_{k=0}^{\infty} \frac{h^k}{k!2^k} H^k\right) = K \otimes K$$

and $\Delta(K^{-1}) = K^{-1} \otimes K^{-1}$. Now

$$\begin{aligned}
&\Delta((q^{1/2} - q^{-1/2})(EF - FE) - K - K^{-1}) = \\
&\qquad = (q^{1/2} - q^{-1/2})[(E \otimes 1 + K \otimes E)(F \otimes K^{-1} + 1 \otimes F) \\
&\qquad\quad - (F \otimes K^{-1} + 1 \otimes F)(E \otimes 1 + K \otimes E)] - K \otimes K + K^{-1} \otimes K^{-1} = \\
&\qquad = (q^{1/2} - q^{-1/2})[(EF - FE) \otimes K^{-1} + K \otimes (EF - FE) \\
&\qquad\quad + KF \otimes EK^{-1} - FK \otimes K^{-1}E] - K \otimes K + K^{-1} \otimes K^{-1} = \\
&\qquad = (q^{1/2} - q^{-1/2})[\frac{1}{q^{1/2} - q^{-1/2}}(K \otimes K^{-1} - K^{-1} \otimes K^{-1} \\
&\qquad\quad + K \otimes K - K \otimes K^{-1}) + KF \otimes EK^{-1} - FK \otimes K^{-1}E] \\
&\qquad\quad - K \otimes K + K^{-1} \otimes K^{-1} = 0
\end{aligned}$$

in the last equality we used that $KF = q^{-1}FK$ and $K^{-1}E = q^{-1}EK^{-1}$ as one can check by direct computation from the definitions. For coassociativity, it suffices to check it on generators and this is computation is analogous to the ones above. □

PROPOSITION 11. *There exists a unique morphism of complete h-adic $\mathbb{Q}[[h]]$-algebras $\epsilon : U_h \to \mathbb{Q}[[h]]$ such that*

$$\epsilon(E) = 0$$
$$\epsilon(F) = 0$$
$$\epsilon(H) = 0$$

furthermore ϵ satisfies the counitality condition.

Proof. This is analogous to the previous proposition. To check compatibility with relations and the counitality condition is useful to note that by continuity of ϵ we have

$$\epsilon(K) = \epsilon\left(\sum_{n=0}^{\infty} \frac{h^n}{n!2^n} H^n\right) = \sum_{n=0}^{\infty} \frac{h^n}{n!2^n} \epsilon(H)^n = 1$$

□

PROPOSITION 12. *There exists a unique antimorphism of complete h-adic $\mathbb{Q}[[h]]$-algebras $S : U_h \to U_h$ such that*

$$\begin{aligned} S(E) &= -K^{-1}E \\ S(F) &= -FK \\ S(H) &= -H \end{aligned}$$

furthermore it satisfies the hexagon condition.

Proof. This is analogous to the propositions above. To check compatibility with relations and hexagon condition is useful to note that by continuity of S we have

$$S(K) = S\left(\sum_{n=0}^{\infty} \frac{h^n}{n!2^n} H^n\right) = \sum_{n=0}^{\infty} \frac{(-1)^n h^n}{n!2^n} H^n = K^{-1}$$

□

These propositions prove that $(U_h, \Delta, \epsilon, S)$ is a Hopf algebra. In [**Hab08**] Habiro gives an R-matrix and a ribbon element that make it a ribbon Hopf algebra

$$v = \exp\left(\frac{h}{2}\right)$$

$$R = \exp\left(\frac{h}{4} H \otimes H\right) \left(\sum_{n=0}^{\infty} v^{n(n-1)/2} \frac{(v - v^{-1})^n}{[n]!} F^n \otimes E^n\right)$$

We conclude this section with some remarks about the structure of U_h that should clarify its relation with the enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ and will be useful later.

Recall that $\mathfrak{sl}_2$ is the Lie algebra (over $\mathbb{Q}$ in our case) whose basis as vector space is given by $\{E, F, H\}$ and bracket defined by

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H$$

Its universal enveloping algebra $U(\mathfrak{sl}_2)$ is the smallest associative $\mathbb{Q}$-algebra with unit in which one can embed $\mathfrak{sl}_2$ as a Lie algebra, where the bracket on $U(\mathfrak{sl}_2)$ is induced by its product $[x, y] = xy - yx$. For precise definitions and background on Lie algebras we refer to [**Hum72**]. We have

$$U_h / hU_h \cong U(\mathfrak{sl}_2)$$

as $\mathbb{Q}$-algebras, an isomorphism is realized by mapping $E \mapsto E$, $F \mapsto F$, $H \mapsto H$ and noticing that the first two defining relations of U_h hold in $U(\mathfrak{sl}_2)$ and for the third we have

$$EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}} = \frac{\sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!2^n} h^n H^n}{\sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!2^n} h^n} =$$

$$= \frac{\frac{hH}{2} \sum_{n=0}^{\infty} \frac{h^{2n}}{(2n+1)!2^{2n}} H^{2n}}{\frac{h}{2} \sum_{n=0}^{\infty} \frac{h^{2n}}{(2n+1)!2^{2n}}} \equiv H \pmod{h}$$

Concerning the module structure, we have

$$U_h^{\otimes m} \cong U(\mathfrak{sl}_2)^{\otimes m}[[h]] \cong S(\mathfrak{sl}_2)^{\otimes m}[[h]]$$

as $\mathbb{Q}[[h]]$-modules, for a detailed explanation of this see [**Kas95**].

3.5 Weight systems

The previous section provides us with an invariant of string links by taking $A = U_h(\mathfrak{sl}_2)$ in the construction of universal quantum invariants, we will denote this invariant simply as $\mathcal{Q}$. From now on we will think at this invariant as valued in $S^{\otimes m}[[h]]$ as described in the end of the previous section. The goal of the present chapter is to prove that a suitable reduction of this invariant is actually a concordance invariant of string links as we describe in Section 3.6.

The approach is to connect $\mathcal{Q}$ to Milnor invariants, in fact these are known to be concordance invariants [**Cas75**] (this also follows from Theorem 3 of Chapter 1). To do this recall that the k-th Milnor invariant has values in $\mathsf{L}_1 \otimes \mathsf{L}_{k+1}$ and thanks to a property called cyclic simmetry [**Mil54, Mil57**] actually in

$$\mathsf{D}_k = \mathrm{Ker}([\cdot,\cdot] : \mathsf{L}_1 \otimes \mathsf{L}_{k+1} \to \mathsf{L}_{k+2})$$

We pointed out in Chapter 1 that there is a group homomorphism turning brackets into trees $\eta_k : \mathcal{T}_k \to \mathsf{D}_k$ and this is not injective in general. On the other hand

$$\eta_k : \mathcal{T}_k \otimes \mathbb{Q} \to \mathsf{D}_k \otimes \mathbb{Q}$$

is an isomorphism of vecor spaces over $\mathbb{Q}$. We can build then a map

$$\mathbb{SL}(m) \underset{\mu_k}{\rightarrow} \mathsf{D}_k \otimes \mathbb{Q} \underset{\eta_k{}^{-1}}{\rightarrow} \mathcal{T}_k \otimes \mathbb{Q} \underset{W}{\rightarrow} S^{\otimes m}[[h]]$$

Here W is a $\mathbb{Q}$-linear map called weight system of the quantum invariant $\mathcal{Q}$, it was introduced by Kontsevich to unify the world of quantum invariants and show that, over the rationals, they all come from a unique invariant called today Kontsevich integral, see for example Chapter 6 of [**Oht02**] for more about this.

The weight system W is defined as follows. If $t \in \mathcal{T}_k \otimes \mathbb{Q}$ connected and $k > 0$, cut in two half edges every edge of t that is adjacent to two degree 3 vertices. This cut gives a collection of k tripodes with labels on the edges and no labels on the half edges, see Figure 3.13.

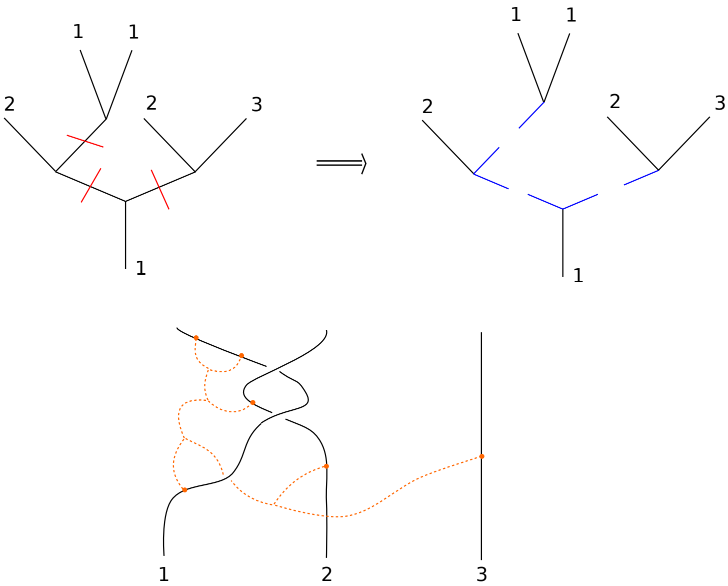

Figure 3.13: Weight system

The Lie bracket of $\mathfrak{sl}_2$ gives a $\mathbb{Q}$-linear map $\mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \to \mathfrak{sl}_2$ and we can use the nondegenerate Killing form of $\mathfrak{sl}_2$

$$k : \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{sl}_2 \quad (x, y) \mapsto \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$$

to identify $\mathfrak{sl}_2 = (\mathfrak{sl}_2)^*$ and represent it as a tensor $J \in \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$, we have denoted here with $\mathrm{ad(x)} : \mathfrak{sl}_2 \to \mathfrak{sl}_2$ the adjoint representation $y \mapsto \mathrm{ad}(x)(y) = [x, y]$ of $\mathfrak{sl}_2$, see [**Hum72**] for more about this. Each tripod inherits a cyclic orientation from t and we number its edges and half edges with $\{1, 2, 3\}$ according to this, with an arbitrary choice of who gets 1. Choose an arbitrary order on tripodes $T_1, \dots, T_k$ and associate to each T_i a 3-tensor $J_i = \epsilon(T_i)J \in \mathfrak{sl}_2^{\otimes 3}$ where $\epsilon(T_i)$ is the sign of t in the vertex of the tripode. Form the product $J_1 \otimes \cdots \otimes J_k \in \mathfrak{sl}_2^{\otimes(a+2b)}$, here a is the number of vertices of degree 1 in t and b is the number of edges that have been cut. Each of the b edges that have been cut is then the union of two half edges with indices $i, j \in \{1, 2, 3\}$ and we contract the i-th component and the j-th component of the corresponding 3-tensors in $J_1 \otimes \cdots \otimes J_k$. After these contractions we get a tensor in $\mathfrak{sl}_2^{\otimes a}$ and each component of this tensor corresponds to a vertex of degree 1 in t and hence has a label in $\{1, \dots, m\}$. We finally get a tensor

$$W_k(t) \in S(\mathfrak{sl}_2)^{\otimes m}$$

whose s-th component is the product in the symmetric algebra $S(\mathfrak{sl}_2)$ of all components in the previous tensor labelled by s. The weight system is finally given by extending W_k to sums of connected trees in $\mathcal{T}_k \otimes \mathbb{Q}$ by $\mathbb{Q}$-linearity and then defining for an arbitrary $t = t_0 + \cdots + t_d \in \mathcal{T} \otimes \mathbb{Q} = \mathcal{T}_0 \otimes \mathbb{Q} \oplus \cdots \oplus \mathcal{T}_d \otimes \mathbb{Q}$

$$W(t) = W_0(t_0)1 + W_1(t_1)h + W_2(t_2)h^2 + \cdots + W_d(t_d)h^d \in S(\mathfrak{sl}_2)^{\otimes m}[[h]]$$

The fact that W is a well-defined function on $\mathcal{T} \otimes \mathbb{Q}$, i.e. that $W(t) = 0$ if $t = 0$ modulo AS and IHX relations defined in Chapter 1, follows form the fact that the Killing form is ad-invariant and the Jacobi identity in $\mathfrak{sl}_2$.

As a final remark, the same construction works using any ad-invariant nondegenerate symmetric bilinear form and not just the Killing form.

3.6 Concordance reduction

We introduce two reduction $\mathbb{Q}$-linear maps $\pi^t, \pi^h : S^{\otimes m}[[h]] \to S^{\otimes m}[[h]]$ defined as follows.

$$\pi^t \left(\sum_{k\geq 0} t_k h^k \right) = \sum_{k\geq 1} (t_k)_{k+1} h^k$$

this map kills the order 0 term of the series and $(\cdot)_{k+1}$ reduces the coefficients of order $k > 0$ taking their degree $k+1$ part in $S^{\otimes m}$, here the grading is given by the tensor grading of the usual grading of the symmetric algebra S, one can think at the latter simply by considering it as the polynomial algebra $\mathbb{Q}[E, F, H]$ and taking as degree the usual degree of multivariate polynomials.

The other is defined as

$$\pi^h \left(\sum_{k\geq 0} t_k h^k \right) = \sum_{k=1}^{m-1} \langle t_k \rangle_{k+1} h^k$$

this maps kills the terms of order 0 and $\geq m$ of the series and $\langle\cdot\rangle_{k+1}$ reduces the coefficients of order $1 \leq k < m$ taking in their degree $k+1$ part the tensors whose degree in each component is 0 or 1.

To simplify notation, we will denote just with μ_k the map

$$\mathbb{SL}(m) \underset{\mu_k}{\to} \mathsf{D}_k \otimes \mathbb{Q} \underset{\eta_k{}^{-1}}{\to} \mathcal{T}_k \otimes \mathbb{Q}$$

and with μ_k^h its reduction modulo trees with no repeated labels on leaves.

The following results are due to Meilhan and Suzuki [**MS14**] and we give here a sketch of their arguments.

THEOREM 7. *If L is a string link with $\mu_k^h(L) = 0$ for $k < N$ then*

$$\pi^h \mathcal{Q}(L) \equiv W \mu_N^h(L) \pmod{h^{N+1}}$$

in particular $\pi^h \mathcal{Q}$ is a link-homotopy invariant of string links.

Proof. The outline of the proof is to reduce the problem to test the congruence on link homotopy classes of string links and then carry out the actual computation on a system of braid representatives for these classes.

The right hand side is known to be invariant up to link-homotopy thanks to the work of Habegger and Lin [**HL90**]. For the invariance of the left hand side we use the theory of claspers, introduce by Habiro in [**Hab00**]. Let L_1, L_2 be link-homotopic string links, then for every $k \geq 1$ there exist $n(k)$ overpassing repeated tree claspers $R_1, \ldots, R_{n(k)}$ of degree $\leq k$ for $\mathbb{1}$ such that

$$L_1 \sim_{C_{k+1}} L_2 \mathbb{1}_{R_1} \cdots \mathbb{1}_{R_{n(k)}}$$

Using that $\mathcal{Q}$ modulo h^{k+1} is a finite type invariant of degree $\leq k$ and the fact that by construction $\mathcal{Q}$ is multiplicative with respect to stacking of string links we have, thanks to the invariance of finite type invariants up to clasper surgeries, that

$$\mathcal{Q}(L_1) \equiv \mathcal{Q}(L_2)\mathcal{Q}(\mathbb{1}_{R_1}) \cdots \mathcal{Q}(\mathbb{1}_{R_{n(k)}}) \pmod{h^{k+1}}$$

The proof of link-homotopy invariance ends with a detailed study of $\mathcal{Q}$ on surgered trivial string links and by taking $k = N$ and using the properties of the map π^h.

The above mentioned result of Habegger and Lin tells that called for $k \geq 2$

$$\mathcal{I}_k = \{i_{\sigma(1)} \cdots i_{\sigma(k-2)} i_{k-1} i_k | 1 \leq i_1 < \cdots < i_k \leq m, \sigma \in S_{k-2}\}$$

then $\{\mu_I | I \in \mathcal{I}_k, 2 \leq k \leq m\}$ is a set of complete link-homotopy invariants of string links (we are thinking here at Milnor invariants as integer numbers).

For every $I \in \mathcal{I}_k$ we have a pure braid (i.e. a string link with monotone strings) defined in terms of commutators

$$B_I = [[\cdots[[A_{i_1 i_2}, A_{i_2 i_3}], A_{i_3 i_4}], \cdots], A_{i_{k-1} i_k}]$$

here A_{st} is the pure braid where the s-th string overpasses the strings $s+1, \ldots, t$ and then underpasses the t-th string and goes back to the s-th position by overpassing all strings, see Figure 3.14 for an example with $m = 4$.

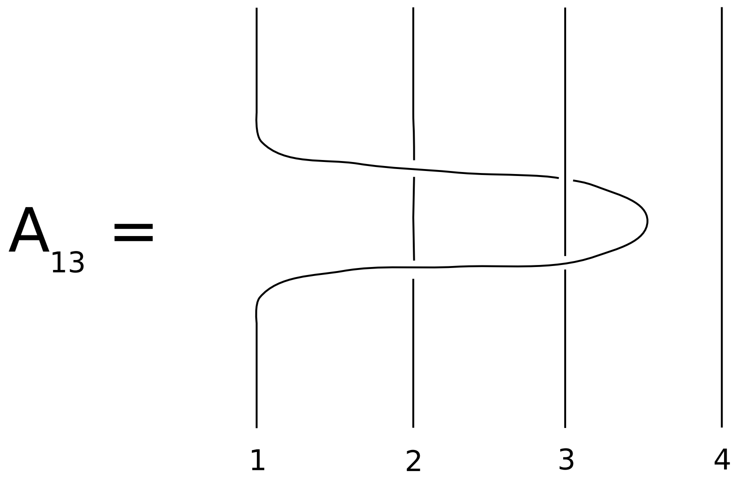

Figure 3.14: Elementary braid

The key point now is that any string link L with m components is link-homotopic to $b_1^L \cdots b_{m-1}^L$ where

$$b_k^L = \prod_{I \in \mathcal{I}_{k+1}} (B_I)^{\mu_I(b_i^L)}$$

here $\mu_I(b_1^L) = \mu_I(L)$ and $\mu_I(b_i^L) = \mu_I(L) - \mu_I(b_1^L \cdots b_{i-1}^L)$ if $i \geq 2$.

The proof of the theorem consists then in checking the claim on this class of string links, for a detailed computation we refer to the original paper of Meilhan and Suzuki. $\square$

THEOREM 8. *If L is a string link with $\mu_k(L) = 0$ for $k < N$ then*

$$\pi^t \mathcal{Q}(L) \equiv W\mu_N(L) \pmod{h^{N+1}}$$

in particular $\pi^t \mathcal{Q}$ is a concordance invariant of string links.

Proof. The outline of the proof is to reduce to the previous theorem via an operation called cabling. For $p \geq 1$ we define the cabling map $D^{(p)} : \mathbb{SL}(m) \to \mathbb{SL}(pm)$ that replaces each component in a string link with p parallel copies, see Figure 3.15 below.

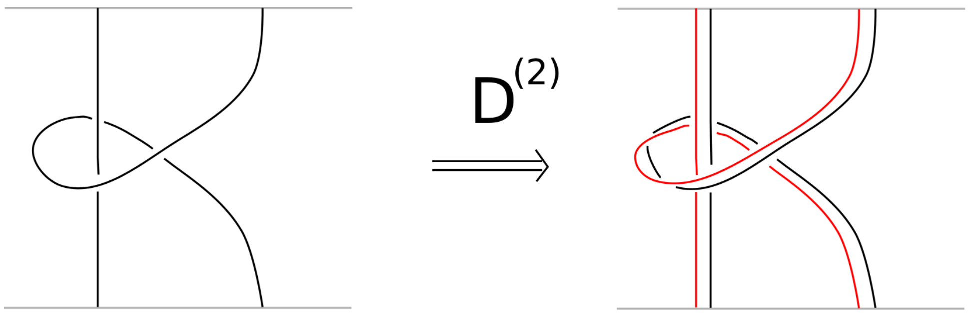

Figure 3.15: Cabling

Now in [**HM00**] is shown that $\mu_k(L) = 0$ for every $k < N$ if and only if $\mu_k^h(D^{(p)}(L)) = 0$ for every $k < N$. The key step is now to observe that for $N < p$ the following diagram commutes

$$\begin{tikzcd}
S^{\otimes m}[[h]] \arrow[d, "\pi^h \Delta_h^{(p)}"'] & \mathbb{SL}_N(m) \arrow[l, "\pi^t \mathcal{Q}"'] \arrow[r, "W\mu_N"] \arrow[d, "D^{(p)}"] & S^{\otimes m}[[h]] \arrow[d, "\pi^h \Delta^{(p)}"] \\
S^{\otimes pm}[[h]] & \mathbb{SL}_N^h(pm) \arrow[l, "\pi^h \mathcal{Q}"] \arrow[r, "W\mu_N^h"'] & S^{\otimes pm}[[h]]
\end{tikzcd}$$

In this diagram $\mathbb{SL}_N$ denotes the set of string links L with $\mu_k(L) = 0$ for $k < N$ and $\mathbb{SL}_N^h$ those with $\mu_k^h(L) = 0$ for $k < N$. On the vertical sides two different coproducts act on $S^{\otimes m}[[h]]$. They are both p-powers of the two coproducts Δ_h and Δ, the first one is induced by the isomorphism of $\mathbb{Q}[[h]]$-modules $U_h^{\otimes m} \cong S^{\otimes m}[[h]]$ and the second one is given by $\Delta(x) = x \otimes 1 + 1 \otimes x$.

For more details we refer to the original paper of Meilhan and Suzuki. The fact that $\pi^t \mathcal{Q}$ is a concordance invariant follows from the concordance invariance of Milnor invariants. $\square$

Bibliography

[Cas75] A. Casson, *Link cobordism and Milnor's invariant*, Bull. London Math. Soc. **7** (1975), 39–40.

[Cim06] D. Cimasoni, *Slicing Bing doubles*, Algebraic & Geometric Topology (2006), no. 6, 2395–2415.

[CST11] J. Conant, R. Schneiderman, and P. Teichner, *Higher-order intersections in low-dimensional topology*, Proceedings of the National Academy of Sciences of the USA **108** (2011), no. 20, 8131–8138.

[CST12] ______, *Whitney tower concordance of classical links*, Geometry & Topology **16** (2012), 1419–1479.

[CST14] ______, *Milnor invariants and twisted Whitney towers*, Journal of Topology **7** (2014), no. 1, 187–224.

[Dri87] V. G. Drinfeld, *Quantum groups*, Proceedings of the ICM, Berkeley 1986, American Mathematical Society, 1987, pp. 798–820.

[Dwy75] W. Dwyer, *Homology, Massey products and maps between groups*, Journal of Pure and Applied Algebra (1975), no. 6, 177–190.

[FM66] R. Fox and J. Milnor, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka J. Math. (1966), no. 3, 257–267.

[FQ90] M. H. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series, vol. 39, Princeton University Press, 1990.

[GS99] R. E. Gompf and A. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, 1999.

[Hab00] K. Habiro, *Claspers and finite type invariants of links*, Geometry & Topology **4** (2000), 1–83.

[Hab08] ______, *A unified Witten-Reshetikhin-Turaev invariant for integral homology spheres*, Inventiones mathematicae **171** (2008), 1–81.

[HL90] N. Habegger and X. Lin, *The classification of links up to link-homotopy*, Journal of the American Mathematical Society **3** (1990), no. 2, 389–419.

[HM00] N. Habegger and G. Masbaum, *The Kontsevich integral and Milnor's invariants*, Topology **39** (2000), no. 6, 1253–1289.

[Hum72] J. E. Humphreys, *Introduction to Lie algebras and Representation Theory*, Graduate Texts in Mathematics, vol. 9, Springer, 1972.

[Jan95] J. C. Jantzen, *Lectures on Quantum Groups*, Graduate Studies in Mathematics, vol. 6, American Mathematical Society, 1995.

[Kas95] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer, 1995.

[Kau87] L. H. Kauffman, *On knots*, Annals of Mathematics Studies, vol. 115, Princeton University Press, 1987.

[KL01] P. Kirk and C. Livingston, *Concordance and mutation*, Geometry & Topology **5** (2001), 831–883.

[Law89] R. J. Lawrence, *A universal link invariant using quantum groups*, Proceedings of the XVII Int. Conf. on Differential Geometric Methods in Theoretical Physics, Chester, England, World Scientific, 1989, pp. 55–63.

[Law90] ______, *A universal link invariant*, Proceedings of the IMA conference on Mathematics-Particle Physics Interface, Oxford, England, Oxford University Press, 1990, pp. 151–156.

[Lev06] J. Levine, *Labeled binary planar trees and quasi-lie algebras*, Algebraic & Geometric Topology (2006), no. 6, 935–948.

[Liv04] C. Livingston, *A survey of classical knot concordance*, Handbook of knot theory (2004), no. 7.

[Mil54] J. Milnor, *Link groups*, Annals of Mathematics **59** (1954), 177–195.

[Mil57] ______, *Isotopy of links*, Algebraic geometry and topology, a symposium in honor of S. Lefschetz, Princeton Mathematical Series, Princeton University Press, 1957, pp. 280–306.

[MS14] J. Meilhan and S. Suzuki, *The universal sl_2 invariant and Milnor invariants*, preprint `http://arxiv.org/abs/1405.3062v1`.

[Oht02] T. Ohtsuki, *Quantum invariants*, Series on Knots and Everything, vol. 29, World Scientific, 2002.

[Orr89] K. Orr, *Homotopy invariants of links*, Inventiones mathematicae (1989), no. 95, 379–394.

[Reu93] C. Reutenauer, *Free Lie algebras*, London Mathematical Society monographs - new series, vol. 7, Oxford Science Publications, 1993.

[Sco05] A. Scorpan, *The wild world of 4-manifolds*, American Mathematical Society, 2005.

[ST04] R. Schneiderman and P. Teichner, *Whitney towers and the Kontsevich integral*, Proceedings of the Casson Fest, Geometry & Topology Monographs, vol. 7, 2004, pp. 101–134.

[Tur90] V. G. Turaev, *Operator invariants of tangles, and R-matrices*, Math. USSR Izvestiya **35** (1990), no. 2.

[War93] S. Warner, *Topological rings*, Mathematics Studies, vol. 178, North-Holland, 1993.